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CHAPTER 5

Trigonometric Equations

Recall that trigonometric equations were introduced in section 1–4, and revisited in 2–6. In this chapter, we first study trigonometric identities; these are very important in the study of the calculus and certain engineering applications. We then examine conditional trigonometric equations in more depth than we did previously.

Section 5–0 reviews important facts about equations and equation solving. Some of this material has been covered in this text, and the rest should be known from previous mathematics courses.

5–0 Equation solving—review

Factoring

To factor means to write as a product. Several common types of factoring that we will need in this chapter to deal with equations are categorized as common factor, difference of two squares, and quadratic trinomial. It is assumed that the student is familiar with each topic. They are presented here both with familiar algebraic forms and with similar trigonometric forms.

	Expression	Factored form
Common factor		
Algebraic	$a^2b - ab^2$	$ab(a - b)$
Trigonometric	$\sin^2\theta \cos\theta - \sin\theta \cos^2\theta$	$\sin\theta \cos\theta(\sin\theta - \cos\theta)$
Difference of two squares		
Algebraic	$a^4 - b^4$	$(a^2 - b^2)(a^2 + b^2)$ or $(a - b)(a + b)(a^2 + b^2)$
Trigonometric	$\sin^4\theta - \cos^4\theta$	$(\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)$ which becomes $(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)(1)$

	Expression	Factored form
Quadratic trinomial		
Algebraic	$2x^2 - x - 3$	$(2x - 3)(x + 1)$
Trigonometric	$2\sin^2 x - \sin x - 3$	$(2\sin x - 3)(\sin x + 1)$

When it is difficult to see how a particular trigonometric expression can be factored, a method called substitution may help. This is illustrated in example 5–0 A.

■ ***Example 5–0 A***

Factor $6\tan^2\theta - 11\tan\theta + 3$

If this is difficult to factor, try substitution.

Let $u = \tan\theta$. Then $u^2 = \tan^2\theta$.

Thus, we can rewrite the equation as

$$6u^2 - 11u + 3$$

which factors into

$$(2u - 3)(3u - 1) \qquad \text{Quadratic trinomial}$$

Now replace u by $\tan\theta$:

$$(2\tan\theta - 3)(3\tan\theta - 1)$$

■

Equations

> An **equation** is a statement of equality of two expressions.

Examples of equations are

$$3x = 27$$
$$3x^2 = 27$$
$$\tan\theta = 1$$
$$x + 3x = 4x$$

An equation involving only one variable is called an equation in one variable.

> A **solution** to an equation in one variable is a real number that makes the statement of equality true when it replaces the variable.

For example, 9 is a solution to the equation $3x = 27$,

3 and -3 are both solutions to the equation $3x^2 = 27$

$\frac{\pi}{4}$ is a solution to the equation $\tan\theta = 1$

any number is a solution to the equation $x + 3x = 4x$

Identities and conditional equations

> An **identity** is an equation that is true for all valid replacement values of the variable.

> A **conditional equation** is an equation that is not true for all valid replacement values of the variable.

The equation

$$3x - 2(x + 5) = x - 10$$

is an identity, which can be seen by combining the terms in the left member, obtaining

$$x - 10 = x - 10$$

The left side will clearly equal the right side regardless of the value of x. We have seen that

$$\sin^2\theta + \cos^2\theta = 1$$

is an identity.

Most of this chapter is concerned with trigonometric identities.

Conditional linear equations

We have solved many linear trigonometric equations in previous sections, like the following.

1. $6 \sin x = 3$
$\sin x = \frac{1}{2}$
$x = \sin^{-1}\frac{1}{2}$

2. $3 \tan \theta - 1 = 1$
$3 \tan \theta = 2$
$\tan \theta = \frac{2}{3}$
$\theta = \tan^{-1}\frac{2}{3}$

Conditional quadratic equations

We have not previously discussed this type of equation. A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. There are two common methods for solving these equations: factoring and the quadratic formula.

Factoring uses the zero product property introduced in section 2–6:

If $ab = 0$, then $a = 0$ or $b = 0$.

The quadratic formula is as follows.

> **Quadratic formula**
> If $ax^2 + bx + c = 0$, and $a \neq 0$, then
> $$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$
> are both solutions to the equation.

The formula is usually abbreviated as $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Both methods for solving quadratic equations are illustrated in example 5–0 B.

■ ***Example 5–0 B***

Solve each quadratic equation for sin x, cos x, or tan x, as appropriate.

1. $6 \tan^2 x - 11 \tan x = -3$

$6 \tan^2 x - 11 \tan x + 3 = 0$ — Add -3 to both members

$(2 \tan x - 3)(3 \tan x - 1) = 0$ — Factor the left member

$2 \tan x - 3 = 0$ or $3 \tan x - 1 = 0$ — Zero product property

$2 \tan x = 3$ or $3 \tan x = 1$

$\tan x = \frac{3}{2}$ or $\tan x = \frac{1}{3}$

2. $2 \cos^2 x - \cos x - 2 = 0$

This expression does not factor, so we use the quadratic formula.

$a = 2, b = -1, c = -2.$

$$\cos x = \frac{-(-1) + \sqrt{(-1)^2 - 4(2)(-2)}}{2(2)} \text{ or } \frac{-(-1) - \sqrt{(-1)^2 - 4(2)(-2)}}{2(2)}$$

$$\cos x = \frac{1 + \sqrt{17}}{4} \text{ or } \frac{1 - \sqrt{17}}{4}$$ ■

Equivalent equations using substitution

Consider the equation

$$(2x - 3)^2 - 3(2x - 3) - 10 = 0$$

If we replace $2x - 3$ by, say u, then we obtain the equation

$$u^2 - 3u - 10 = 0$$

which we solve as

$$(u - 5)(u + 2) = 0$$
$$u = -2 \text{ or } 5$$

We now replace u by $2x - 3$ to obtain

$$2x - 3 = -2 \text{ or } 2x - 3 = 5$$
$$2x = 1 \text{ or } 2x = 8$$
$$x = \tfrac{1}{2} \text{ or } x = 4$$

Anytime we can replace an expression by a variable like u we obtain an equivalent equation that may be easier to solve or otherwise manipulate.

■ ***Example 5–0 C***

Use substitution to help solve each problem.

1. Solve the equation $2(5x + 3)^3 - (5x + 3)^2 = 0$.
Let $u = 5x + 3$. Then,

$$2u^3 - u^2 = 0$$
$$u^2(2u - 1) = 0$$
$$u^2 = 0 \text{ or } 2u - 1 = 0$$
$$u = 0 \text{ or } u = \tfrac{1}{2}$$
$$5x + 3 = 0 \text{ or } 5x + 3 = \tfrac{1}{2}$$
$$5x = -3 \text{ or } 5x = -\tfrac{5}{2} \qquad \tfrac{1}{2} - 3 = \tfrac{1}{2} - \tfrac{6}{2}$$
$$x = -\tfrac{3}{5} \text{ or } x = \tfrac{1}{5}(-\tfrac{5}{2}) = -\tfrac{1}{2}$$

2. Simplify $\dfrac{\left(3\theta - \dfrac{\pi}{2}\right)^2 - 1}{1 - \left(3\theta - \dfrac{\pi}{2}\right)}$.

Let $u = 3\theta - \dfrac{\pi}{2}$. Then the expression is

$$\frac{u^2 - 1}{1 - u} = \frac{(u - 1)(u + 1)}{-(u - 1)} = \frac{u + 1}{-1} = -(u + 1) = -u - 1$$

Replace u by $3\theta - \dfrac{\pi}{2}$:

$$-u - 1 = -\left(3\theta - \frac{\pi}{2}\right) - 1 = -3\theta + \frac{\pi}{2} - 1$$ ■

Mastery points

Can you
- Factor trigonometric expressions?
- Solve quadratic equations for *x*, sin *x*, cos *x*, or tan *x*?
- Use substitution to help solve equations and simplify expressions?

Exercise 5–0

Note: The *solutions* to all of these exercises are given in appendix E. This section is not reviewed explicitly in the chapter review or the chapter test. It is designed to prepare for the rest of the chapter.

Factor each trigonometric expression.

1. $\sin^2\theta - \sin\theta$
2. $\cos^3\theta + 3\cos\theta$
3. $\cos^4\theta - \cos^2\theta$
4. $\sin^5\theta - \sin^3\theta$
5. $\cos^2 x + \cos x - 20$
6. $\tan^2 x + 2\tan x - 24$
7. $2\sin^2 x - 7\sin x + 3$
8. $9\cos^3\theta - 15\cos^2\theta - 6\cos\theta$
9. $6\csc^2\theta - 5\csc\theta + 1$

Solve each quadratic equation for sin θ, cos θ, tan θ, etc., as appropriate.

10. $\tan^2\theta - \tan\theta = 0$

11. $\tan^2\theta - \tan\theta = 2$

12. $6\sin^2\theta + 5\sin\theta + 1 = 0$

13. $\sec^4\theta - 5\sec^2\theta + 4 = 0$

14. $36\sin^4\theta - 13\sin^2\theta + 1 = 0$

15. $\frac{2}{3}\sin\theta + \frac{1}{3\sin\theta} = 1$

16. $\sin^2\theta - 3\sin\theta - 5 = 0$

17. $2\sec^2\theta + 3\sec\theta - 7 = 0$

Use substitution to help solve each equation.

18. $(2x - 6)^3 - (2x - 6)^2 = 0$

19. $12\left(\frac{x}{3} - 1\right)^2 - 5\left(\frac{x}{3} - 1\right) - 2 = 0$

Use substitution to help simplify each expression.

20. $5\left(\frac{\pi}{2} - 3\right) + 3\left[\left(\frac{\pi}{2} - 3\right) - 7\right] - \frac{\pi}{2} + 3$

21. $\dfrac{2(3x - \frac{1}{2})^2 - 3(3x - \frac{1}{2}) + 1}{(3x - \frac{1}{2})^2 - 1}$

5–1 Basic trigonometric identities

Review of some identities

Recall from section 1–4 and 5–0 that an identity is an equation that is true for every allowed value of its variable (or variables). We have seen the following identities in previous sections.

Reciprocal identities

$\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{1}{\tan\theta}$

$\sin\theta = \frac{1}{\csc\theta}$, $\cos\theta = \frac{1}{\sec\theta}$, $\tan\theta = \frac{1}{\cot\theta}$

Tangent/cotangent identities

$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Fundamental identity of trigonometry

$\sin^2\theta + \cos^2\theta = 1$

Two other forms of the fundamental identity are $\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$.

If each term in the fundamental identity is divided by $\cos^2\theta$ we obtain

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2$$

$$\tan^2\theta + 1 = \sec^2\theta$$

Similarly, if each term of the fundamental identity is divided by $\sin^2\theta$ we obtain the identity $\cot^2\theta + 1 = \csc^2\theta$. These two identities, along with the fundamental identity, are called the Pythagorean identities. They are summarized here.

Pythagorean identities

	Useful forms	
$\sin^2\theta + \cos^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$	$\cos^2\theta = 1 - \sin^2\theta$
$\tan^2\theta + 1 = \sec^2\theta$	$\tan^2\theta = \sec^2\theta - 1$	$\sec^2\theta - \tan^2\theta = 1$
$\cot^2\theta + 1 = \csc^2\theta$	$\cot^2\theta = \csc^2\theta - 1$	$\csc^2\theta - \cot^2\theta = 1$

Preliminary notes on algebra

When working with trigonometric equations there are certain algebraic principles that are used over and over. It is a good idea to get used to these principles, and the associated notation, now—it will make the rest of this chapter much easier.

These principles are illustrated in example 5–1 A. We also use the principle that we *do not leave fractions in a final answer when possible.* For example, $\dfrac{1}{\cot\theta}$ can be rewritten as $\tan\theta$. Also we note that a binomial of the form $x + y$ is called the **conjugate** of the binomial $x - y$, and vice versa.

■ Example 5–1 A

Perform the algebra indicated, and note the category of algebraic manipulation for later reference. Do not leave a fraction for an answer when possible.

1. *Separating fractions:* $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$

   Rewrite $\dfrac{1-\sin\theta}{\sin\theta}$ as two fractions.

$$\frac{1-\sin\theta}{\sin\theta} = \frac{1}{\sin\theta} - \frac{\sin\theta}{\sin\theta}$$
$$= \csc\theta - 1$$

2. *Multiplying binomial conjugates:* $(a + b)(a - b) = a^2 - b^2$
   Multiply $(1 - \sin\theta)(1 + \sin\theta)$.

$$(1 - \sin\theta)(1 + \sin\theta) = 1^2 - (\sin\theta)^2$$
$$= 1 - \sin^2\theta$$
$$= \cos^2\theta \quad \text{Pythagorean identity}$$

3. *Factoring quadratic binomials into conjugates:* $a^2 - b^2 = (a + b)(a - b)$

   Factor the numerator of $\dfrac{\csc^2\theta - \cot^2\theta}{\csc\theta - \cot\theta}$.

$$\frac{\csc^2\theta - \cot^2\theta}{\csc\theta - \cot\theta} = \frac{(\csc\theta - \cot\theta)(\csc\theta + \cot\theta)}{\csc\theta - \cot\theta}$$
$$= \csc\theta + \cot\theta$$

4. *Multiplying numerator and denominator of a fraction by a conjugate of the numerator or denominator:*

   Multiply the numerator and denominator of $\dfrac{1 - \cos\theta}{\sin\theta}$ by the conjugate of the numerator.

$$\begin{aligned} \frac{1 - \cos\theta}{\sin\theta} \cdot \frac{1 + \cos\theta}{1 + \cos\theta} &= \frac{1 - \cos^2\theta}{\sin\theta(1 + \cos\theta)} \\ &= \frac{\sin^2\theta}{\sin\theta(1 + \cos\theta)} \\ &= \frac{\sin\theta}{1 + \cos\theta} \end{aligned}$$

5. *Factoring the sign from a binomial:* $-(a - b) = b - a$

   Simplify $\dfrac{\cos^2\theta - 1}{\sin\theta}$.

$$\begin{aligned} \frac{\cos^2\theta - 1}{\sin\theta} &= \frac{-(1 - \cos^2\theta)}{\sin\theta} \\ &= \frac{-\sin^2\theta}{\sin\theta} \\ &= -\sin\theta \end{aligned}$$

Transforming expressions

Identities and the principles illustrated above aid in simplifying and transforming trigonometric expressions. We proceed by replacing given parts of an expression by equivalent parts from the identities summarized above, as well as any other identities we have studied. *There is no single correct sequence of steps!* We proceed by trial and error, guided by past experience.

Although there are many ways to proceed in transforming an expression, we will note some guidelines for this process.

1. When functions appear raised to the second power, such as $\sin^2\theta$, $\tan^2\theta$, etc., look for expressions that appear in the Pythagorean identities.
   a. It may be possible to combine two terms into one.

   example: $\dfrac{\sec^2\theta - \tan^2\theta}{\sin\theta}$ becomes $\dfrac{1}{\sin\theta}$, which becomes $\csc\theta$.

   b. It is always possible to rewrite one second degree term as two.

   example: $\dfrac{\sec^2\theta}{\tan^2\theta + 1}$ becomes $\dfrac{\tan^2\theta + 1}{\tan^2\theta + 1}$, which becomes 1.

2. Sometimes it pays to rewrite all functions in terms of the sine and cosine functions.

   Rewrite $\sec\theta$ as $\dfrac{1}{\cos\theta}$, $\csc\theta$ as $\dfrac{1}{\sin\theta}$, $\tan\theta$ as $\dfrac{\sin\theta}{\cos\theta}$, $\cot\theta$ as $\dfrac{\cos\theta}{\sin\theta}$.

3. Look for factors of the Pythagorean identities. These are things like $1 - \cos\theta$, which is a factor of $1 - \cos^2\theta$, and $\sec\theta + \tan\theta$, which is a factor of $\sec^2\theta - \tan^2\theta$. These can often be transformed as in part 4 of example 5–1 A.

   Example 5–1 B illustrates these guidelines.

■ ***Example 5–1 B***

Simplify each expression into one term.

1. $1 - \cos^2 4\alpha$

$\sin^2 4\alpha$ — $\sin^2 4\alpha + \cos^2 4\alpha = 1$, so $1 - \cos^2 4\alpha = \sin^2 4\alpha$

2. $(1 - \sec x)(1 + \sec x)$

$1 - \sec^2 x$ — $(a - b)(a + b) = a^2 - b^2$

$-(\sec^2 x - 1)$

$-\tan^2 x$

Simplify the expression into as few factors as possible.

3. $\cot\theta\sec\theta\sin\theta$

$\cot\theta\sec\theta\sin\theta$

$\dfrac{\cos\theta}{\sin\theta}\cdot\dfrac{1}{\cos\theta}\cdot\sin\theta$ — Rewrite everything in terms of sin θ, cos θ

$\dfrac{\overset{1}{\cancel{\cos\theta}}}{\underset{1}{\cancel{\sin\theta}}}\cdot\dfrac{1}{\underset{1}{\cancel{\cos\theta}}}\cdot\overset{1}{\cancel{\sin\theta}}$ — Divide out the common factors

1 ■

Verifying identities

If we can show that one member of an equation can be transformed into the other member by replacing expressions using identities and performing algebraic transformations, then we say we have *verified* the equation to be an identity.

Although there are many ways to proceed in verifying an identity, there are some guidelines for this process. The guidelines 1 through 3 apply to verifying identities as well as simplifying individual expressions. Example 5–1 C illustrates another guideline, which applies to verifying identities.

4. Begin with the more complicated member of an equation, and try to simplify it.

A fraction is almost always considered more complicated than a nonfraction.

■ ***Example 5–1 C***

Verify that each equation is an identity by showing that one member of the identity can be transformed into the other member.

1. $\tan\theta\csc\theta = \sec\theta$

$\tan\theta\csc\theta$	Begin with the left member
$\dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{1}{\sin\theta}$	Rewrite everything in terms of $\sin\theta$ and $\cos\theta$
$\dfrac{1}{\cos\theta}$	Reduce by a factor of $\sin\theta$
$\sec\theta$	$\sec\theta = \dfrac{1}{\cos\theta}$

2. $\dfrac{1}{\sin\beta - \csc\beta} = -\tan\beta\sec\beta$

$\dfrac{1}{\sin\beta - \csc\beta}$	Begin with the left member; it is more complicated
$\dfrac{1}{\sin\beta - \dfrac{1}{\sin\beta}}$	Rewrite everything in terms of $\sin\theta$ and $\cos\theta$; we arrive at a complex fraction
$\dfrac{1}{\sin\beta - \dfrac{1}{\sin\beta}}\cdot\dfrac{\sin\beta}{\sin\beta}$	Multiply numerator and denominator by $\sin\beta$; this is to simplify the complex fraction
$\dfrac{\sin\beta}{\sin^2\beta - 1}$	$\sin\beta\left(\sin\beta - \dfrac{1}{\sin\beta}\right) = \sin^2\beta - 1$
$\dfrac{\sin\beta}{-(1-\sin^2\beta)}$	Factor the sign from the binomial denominator
$\dfrac{\sin\beta}{-\cos^2\beta}$	
$-\dfrac{\sin\beta}{\cos\beta}\cdot\dfrac{1}{\cos\beta}$	
$-\tan\beta\sec\beta$	

3. $\dfrac{\cos^2\alpha}{1+\sin\alpha} = 1 - \sin\alpha$

$\dfrac{\cos^2\alpha}{1+\sin\alpha}$	
$\dfrac{1-\sin^2\alpha}{1+\sin\alpha}$	$\cos^2\theta = 1 - \sin^2\theta$
$\dfrac{(1-\sin\alpha)(1+\sin\alpha)}{1+\sin\alpha}$	$m^2 - n^2 = (m-n)(m+n)$
$1 - \sin\alpha$	

■

It is not absolutely necessary to transform just one side or the other of an identity. When both members of an identity are very complicated it is easier to transform both members, as in example 5–1 D.

■ Example 5–1 D

Verify the identity.

$$\sin^2\theta\tan^2\theta + 1 = \sec^2\theta - \cos^2\theta\sec^2\theta + \cos^2\theta$$

Left member	**Right member**
$\sin^2\theta\tan^2\theta + 1$	$\sec^2\theta - \cos^2\theta\sec^2\theta + \cos^2\theta$
$\sin^2\theta\tan^2\theta + \sin^2\theta + \cos^2\theta$	$\sec^2\theta(1 - \cos^2\theta) + \cos^2\theta$
$\sin^2\theta(\tan^2\theta + 1) + \cos^2\theta$	$\sec^2\theta\sin^2\theta + \cos^2\theta$
$\sin^2\theta\sec^2\theta + \cos^2\theta$	$\dfrac{1}{\cos^2\theta}\sin^2\theta + \cos^2\theta$
$\sin^2\theta\dfrac{1}{\cos^2\theta} + \cos^2\theta$	
$\tan^2\theta + \cos^2\theta$	$\tan^2\theta + \cos^2\theta$ ■

Since the left side and right side can be transformed into the same expression, they are equivalent. This is true because we could actually take the steps on one side and add them to the other side in reverse order, arriving at the required result.

Of course most equations are not identities. To show that an equation is not an identity we need to find a value for the variable for which each member of the equation is defined, but that produces different results in the two members. This value is called a **counter example;** it shows that the equation is *not* an identity.

■ Example 5–1 E

Show by counter example that $\cos x + \sin x \cot x = \sin x$ is not an identity.

Choose a value for which both the left and right members are defined; $x = \dfrac{\pi}{4}$ is such a value (most values would serve the purpose).

Left member	**Right member**	
$\cos x + \sin x \cot x$	$\sin x$	
$\cos\dfrac{\pi}{4} + \sin\dfrac{\pi}{4}\cot\dfrac{\pi}{4}$	$\sin\dfrac{\pi}{4}$	Replace x by $\dfrac{\pi}{4}$
$\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\cdot 1$	$\dfrac{\sqrt{2}}{2}$	$\cos\dfrac{\pi}{4} = \sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$; $\cot\dfrac{\pi}{4} = 1$
$\dfrac{2\sqrt{2}}{2}$		
$\sqrt{2}$		

Since $\sqrt{2} \neq \dfrac{\sqrt{2}}{2}$ we have shown that the given equation is not an identity. ■

As noted in example 5–1 E, most values will serve as a counter example for an equation that is not an identity. However, *avoid using zero,* since there are many equations that are not identities for which both sides evaluate to the same value when zero is used.

An example is $\sin\theta = 1 - \cos\theta$. This equation is *not* an identity, but observe that replacing θ by 0 produces $0 = 0$, which might lead one to believe that this equation is an identity.

Mastery points

Can you

- State the reciprocal identities, fundamental identity, and the remaining Pythagorean identities from memory?
- Recognize useful forms of the Pythagorean identities?
- Transform forms of the Pythagorean identities into simpler forms?
- Transform one side of an identity into the other side?
- Show that an equation is not an identity by a counter example?

Exercise 5–1

Each of the following expressions can be simplified into the form 1, -1, $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$, $\sin^2\theta$, $\cos^2\theta$, $\tan^2\theta$, $\cot^2\theta$, $\sec^2\theta$, or $\csc^2\theta$. Show the transformation of each expression into one of these forms.

1. $\dfrac{\sin\theta}{\tan\theta}$
2. $\dfrac{\cos\theta}{\cot\theta}$
3. $\tan\theta\csc\theta$
4. $\sec\theta\cot\theta$
5. $\cot^2\theta\sin^2\theta$
6. $\sin^2\theta\sec^2\theta$
7. $(\tan^2\theta + 1)(1 - \sin^2\theta)$
8. $(1 - \cos^2\theta)(1 + \cot^2\theta)$
9. $\dfrac{(\sec\theta - 1)(\sec\theta + 1)}{\sin^2\theta}$
10. $\dfrac{(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)}{\cos^2\theta}$
11. $\dfrac{\csc\theta\sin\theta}{\cot\theta}$
12. $\dfrac{\tan\theta\cot\theta}{\sin\theta}$
13. $\cos\theta(\sec\theta - \cos\theta)$
14. $\cos^2\theta(1 + \cot^2\theta)$
15. $\csc^2\theta(1 - \cos^2\theta)$
16. $\sin^2\theta(\csc^2\theta - 1)$
17. $\cot^2\theta - \csc^2\theta$
18. $\tan^2\theta - \sec^2\theta$
19. $\tan^2\theta(\cot^2\theta + 1)$
20. $\dfrac{\cot x}{\sec x}$
21. $\sec^2\theta(\csc^2\theta - 1)$
22. $\dfrac{\sec\theta}{\tan\theta\csc\theta}$
23. $\dfrac{\cot\theta\sec\theta}{\csc\theta}$
24. $\dfrac{\csc^2\theta - 1}{\csc^2\theta}$
25. $\sin x + \cos x\cot x$
26. $\cos x + \sin x\tan x$
27. $\tan x\csc x\cos x$
28. $\dfrac{\csc x + \sec x}{\tan x + 1}$
29. $\sec x - \tan x\sin x$
30. $(\csc x + 1)(\sec x - \tan x)$
31. $(\csc x + \cot x)(1 - \cos x)$
32. $\dfrac{\sec^4 y - \tan^4 y}{\sec^2 y + \tan^2 y}$

Verify the following identities.

33. $\csc\theta + \cot\theta = \dfrac{1+\cos\theta}{\sin\theta}$

34. $\tan\theta + \sec\theta = \dfrac{1+\sin\theta}{\cos\theta}$

35. $\dfrac{\csc\theta}{\sec\theta+\tan\theta} = \dfrac{\cos\theta}{\sin\theta+\sin^2\theta}$

36. $\dfrac{\sec\theta}{\csc\theta+\cot\theta} = \dfrac{\tan\theta}{1+\cos\theta}$

37. $\dfrac{1+\csc\theta}{1+\sec\theta} = \cot\theta\left(\dfrac{1+\sin\theta}{1+\cos\theta}\right)$

38. $\dfrac{\tan\theta+1}{\tan\theta-1} = \dfrac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}$

39. $\dfrac{\tan^2\theta+\sec^2\theta}{\sec^2\theta} = \sin^2\theta + 1$

40. $\dfrac{\cot^2\theta+\csc^2\theta}{\csc^2\theta} = 1 + \cos^2\theta$

41. $\tan y \sin y = \sec y - \cos y$

42. $\cot x \cos x = \csc x - \sin x$

43. $\dfrac{1+\cot^2\theta}{\tan^2\theta} = \cot^2\theta\csc^2\theta$

44. $\dfrac{\cot^2\theta}{\csc\theta+1} = 1 - \csc\theta$

45. $\dfrac{1}{\sec\theta-\cos\theta} = \cot\theta\csc\theta$

46. $\dfrac{1}{\cot\theta+\tan\theta} = \sin\theta\cos\theta$

47. $\dfrac{\tan\theta}{\csc\theta} = \sin\theta\tan\theta$

48. $\dfrac{\cot\theta}{\sec\theta} = \csc\theta - \sin\theta$

49. $\dfrac{\tan^2\theta}{\sec\theta-1} = 1 + \sec\theta$

50. $\dfrac{1}{\sec x-\tan x} = \sec x + \tan x$

51. $\dfrac{\cot x+1}{\cot x-1} = \dfrac{\sin x+\cos x}{\cos x-\sin x}$

52. $\dfrac{1+\sin y}{1-\sin y} = \dfrac{\csc y+1}{\csc y-1}$

53. $\dfrac{\sin x}{1+\cos x} = \dfrac{1-\cos x}{\sin x}$

54. $\dfrac{1+\sin y}{\cos y} = \dfrac{\cos y}{1-\sin y}$

55. $\dfrac{\cos x}{\sec x-\tan x} = \dfrac{\cos^2 x}{1-\sin x}$

56. $\dfrac{\cos x}{\cos x+\sin x} = \dfrac{\cot x}{1+\cot x}$

57. $\dfrac{1}{1-\sin x} + \dfrac{1}{1+\sin x} = 2\sec^2 x$

58. $\dfrac{\cos y}{\csc y+1} + \dfrac{\cos y}{\csc y-1} = 2\tan y$

59. $\sin^4 x - \cos^4 x = 1 - 2\cos^2 x$

60. $(\tan^2 y + 1)(\cot^2 y + 1) = \sec^2 y + \csc^2 y$

61. $\cot^2 x - \cos^2 x = \cot^2 x\cos^2 x$

62. $\csc^2 y + \sec^2 y = \sec^2 y\csc^2 y$

63. $\dfrac{\tan y-\cot y}{\tan y+\cot y} = \dfrac{\tan^2 y-1}{\sec^2 y}$

64. $\dfrac{\cot^2 x-1}{\cot^2 x+1} = 1 - 2\sin^2 x$

65. $\dfrac{1-\sin x}{1+\sin x} = (\tan x - \sec x)^2$

66. $\sec^4 x - 1 = \tan^2 x\sec^2 x + \tan^2 x$

67. $\sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x$

68. $\tan x - \cot x = \dfrac{\sin^2 x-\cos^2 x}{\sin x\cos x}$

69. $2\cos^2 y - 1 = \cos^2 y - \sin^2 y$

70. $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$

In problems 71–80 show by counter example that each equation is not an identity.

71. $\sin\theta = 1 - \cos\theta$

72. $\tan^2\theta - \cot^2\theta = 1$

73. $\sec\theta = \dfrac{1}{\csc\theta}$

74. $\sin\theta = \dfrac{1}{\cos\theta}$

75. $\sin^2\theta - 2\cos\theta\sin\theta + \cos^2\theta = 2$

76. $\tan^2\theta - \tan\theta = 0$

77. $\csc\theta + \sec\theta\cot\theta = 2$

78. $\sin\theta + 2\sin\theta\cos\theta = 0$

79. $\dfrac{1-\cos\theta}{1+\cos\theta} = \sin^2\theta$

80. $\dfrac{1}{\tan\theta+\csc\theta} = \sec\theta$

81. Verify by calculation that $(1 - \csc^2\theta)(1 - \sec^2\theta) = 1$ for the values

a. $\theta = \dfrac{\pi}{6}$ **b.** $\theta = \dfrac{\pi}{4}$ **c.** Is this equation an identity?

82. Verify by calculation that $\dfrac{\sin\theta-\cos\theta}{\cos\theta} = \tan\theta - 1$ for the values

a. $\theta = \dfrac{\pi}{3}$ **b.** $\theta = \dfrac{3\pi}{4}$ **c.** Is this equation an identity?

83. Verify by calculation that $2\sin^2\theta + \sin\theta = 1$ for the values

a. $\theta = \dfrac{\pi}{6}$ **b.** $\theta = \dfrac{3\pi}{2}$ **c.** Is this equation an identity?

84. Verify by calculation that $\tan^4\theta - \tan^2\theta = 6$ for the values

a. $\theta = \dfrac{\pi}{3}$ **b.** $\theta = \dfrac{4\pi}{3}$ **c.** Is this equation an identity?

5–2 The sum and difference identities

Four important identities are called the sum and difference identities.

Sum and difference identities for sine and cosine

[1] $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

[2] $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

[3] $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

[4] $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

The last three of these four identities can be developed using the first. Their verification is left as exercises. A demonstration that identity [1] is true is given in appendix B.

The sum and difference identities have several applications, illustrated in example 5–2 A.

■ *Example 5–2 A*

1. Use the fact that $\frac{7\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$ to find the exact value of $\cos\frac{7\pi}{12}$.

$$\cos\frac{7\pi}{12} = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$

$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} \qquad \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}$$

2. Show that $\cos(\pi - \theta) = -\cos\theta$ for any angle θ.

$$\begin{aligned}\cos(\pi - \theta) &= \cos\pi\cos\theta + \sin\pi\sin\theta \\ &= (-1)\cos\theta + 0\sin\theta \\ &= -\cos\theta\end{aligned}$$

■

Identity [1] can be used to prove the following identities (the proofs are left for the exercises). These identities are called the cofunction identities.

Cofunction identities

[5]	$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$	[6]	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$
[7]	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$	[8]	$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$
[9]	$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$	[10]	$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

The reason for the name of these identities is as follows.

When the sum of two angles is 90°, or $\frac{\pi}{2}$ radians, the angles are said to be **complementary.** The angles $\frac{\pi}{2} - \theta$ and θ add up to $\frac{\pi}{2}$, so they are complementary angles. Each is said to be the complement of the other. The cofunction identities say, in effect,

trig function (angle) = ''co'' trig function (complement of angle)

Thus, the sine and ''co'' sine appear in one identity, the tangent and ''co'' tangent appear in another, and the secant and ''co'' secant in the third. Whenever the sum of two angles is $\frac{\pi}{2}$ (or 90°), a trigonometric function of one equals the ''co'' trigonometric function of the other. Thus for example, the following statements are true:

$$\sin 50° = \cos 40° \qquad 50° + 40° = 90°$$

$$\sec \frac{\pi}{6} = \csc \frac{\pi}{3} \qquad \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$$

$$\cot 130° = \tan(-40°) \qquad 130° + (-40°) = 90°$$

■ *Example 5–2 B*

Rewrite each function value in terms of its cofunction.

1. $\sin 34°$

$$\sin 34° = \cos(90° - 34°) = \cos 56°$$

2. $\csc \frac{2\pi}{5}$

$$\csc \frac{2\pi}{5} = \sec\left(\frac{\pi}{2} - \frac{2\pi}{5}\right) = \sec \frac{\pi}{10}$$

Simplify each expression.

3. $\frac{\sin 10°}{\cos 80°}$

$$\frac{\sin 10°}{\cos 80°} = \frac{\cos 80°}{\cos 80°} = 1$$

4. $\sin^2\frac{\pi}{6} + \sin^2\frac{\pi}{3}$

$$\sin^2\frac{\pi}{6} + \sin^2\frac{\pi}{3} = \sin^2\frac{\pi}{6} + \cos^2\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{6} = 1 \quad ■$$

Two more important identities are the sum and difference formulas for the tangent function.

Sum and difference identities for tangent

$$[11] \quad \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$[12] \quad \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

The derivation of the first identity is as follows.

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}$$

$$= \frac{\dfrac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\dfrac{\cos \alpha \cos \beta - \sin \alpha \sin \beta}{\cos \alpha \cos \beta}}$$

Divide numerator and denominator by $\cos \alpha \cos \beta$

$$= \frac{\dfrac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \dfrac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\dfrac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} - \dfrac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}}$$

$$= \frac{\dfrac{\sin \alpha}{\cos \alpha} + \dfrac{\sin \beta}{\cos \beta}}{1 - \dfrac{\sin \alpha}{\cos \alpha} \cdot \dfrac{\sin \beta}{\cos \beta}} = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

Example 5–2 C illustrates using this identity.

■ ***Example 5–2 C***

Use the fact that $15° = 45° - 30°$ to find the exact value of $\tan 15°$.

$$\tan 15° = \tan(45° - 30°)$$

$$= \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} = \frac{1 - \dfrac{\sqrt{3}}{3}}{1 + 1\left(\dfrac{\sqrt{3}}{3}\right)}$$

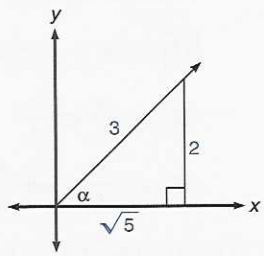
$$= \frac{1 - \dfrac{\sqrt{3}}{3}}{1 + \dfrac{\sqrt{3}}{3}} \cdot \frac{3}{3} = \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}}$$

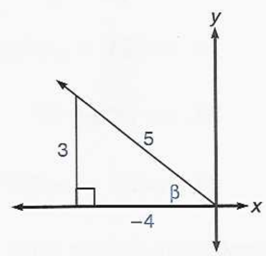
$$= \frac{9 - 6\sqrt{3} + 3}{9 - 3} = \frac{12 - 6\sqrt{3}}{6} = 2 - \sqrt{3}$$ ■

Some problems can be solved by using the identities above and reference triangles (section 2–4).

■ ***Example 5–2 D***

$\sin \alpha = \frac{2}{3}$, α in quadrant I; $\cos \beta = -\frac{4}{5}$, β in quadrant II. Find the exact value of $\cos(\alpha - \beta)$.





$$\cos \alpha = \frac{\sqrt{5}}{3}, \sin \beta = \frac{3}{5}$$

We find the necessary values from the reference triangles

$$\begin{aligned}\cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ &= \frac{\sqrt{5}}{3} \cdot \left(-\frac{4}{5}\right) + \frac{2}{3} \cdot \frac{3}{5} \\ &= -\frac{4\sqrt{5}}{15} + \frac{6}{15} \\ &= \frac{-4\sqrt{5} + 6}{15}\end{aligned}$$

■

Mastery points

Can you

- State the sum and difference identities?
- State and apply the cofunction identities?
- Apply the sum and difference identities to find exact values of sine, cosine, and tangent for certain angles?
- Apply the sum and difference identities to find exact values of sine, cosine, and tangent for $\alpha + \beta$ and $\alpha - \beta$ given information about α and β?
- Verify identities using the sum and difference identities?

Exercise 5–2

Rewrite each function in terms of its cofunction.

1. $\sin 18°$ **2.** $\cos 42°$ **3.** $\tan 8°$ **4.** $\csc 100°$ **5.** $\sec \frac{\pi}{3}$

6. $\cot \frac{\pi}{6}$ **7.** $\cos \frac{5\pi}{6}$ $\boxed{\textbf{8.}}$ $\sin\left(-\frac{\pi}{3}\right)$ **9.** $\sec\left(-\frac{3\pi}{4}\right)$ **10.** $\csc\left(-\frac{\pi}{4}\right)$

Simplify each expression.

11. $\dfrac{\cos 65°}{\sin 25°}$
12. $\tan\dfrac{\pi}{3}\tan\dfrac{\pi}{6}$
13. $\cos 20° \csc 70°$
14. $\dfrac{\sin^2 5°}{\cos^2 85°}$
15. $\sin\dfrac{\pi}{5}\sec\dfrac{3\pi}{10}$
16. $\cos^2 25° + \cos^2 65°$
17. $\tan^2 8° - \csc^2 82°$
18. $\dfrac{\cos^2 30°}{1 - \cos^2 60°}$
19. $\tan 40° \tan 50°$
20. $\tan 19° \tan 71°$
21. $\sec\dfrac{\pi}{6}\sin\dfrac{\pi}{3}$
22. $\cot\dfrac{\pi}{5}\cot\dfrac{3\pi}{10}$
23. $\sin^2 10° + \sin^2 80°$
24. $\tan^2 25° - \csc^2 65°$
25. $\sec^2\dfrac{\pi}{3} - \cot^2\dfrac{\pi}{6}$
26. $\cos^2\dfrac{3\pi}{8} + \cos^2\dfrac{\pi}{8}$

Use the sum and difference identities to find the exact value of each of the following. Observe that each value is the sum or difference of values chosen from $\dfrac{\pi}{6}(30°)$, $\dfrac{\pi}{4}(45°)$, and $\dfrac{\pi}{3}(60°)$.

27. $\cos\dfrac{\pi}{12}$
28. $\tan\dfrac{\pi}{12}$
29. $\sin\dfrac{5\pi}{12}$
30. $\cos\dfrac{5\pi}{12}$
31. $\sin\dfrac{7\pi}{12}$
32. $\tan\dfrac{7\pi}{12}$
33. $\sin 15°$
34. $\tan 15°$
35. $\cos 105°$
36. $\sin 105°$
37. $\tan 75°$
38. $\cos 15°$

Each of the following problems presents information about two angles, α and β, including the quadrant in which the angle terminates. Use the information to find the required value.

39. $\cos\alpha = \frac{1}{3}$, quadrant I; $\sin\beta = \frac{3}{4}$, quadrant I. Find $\sin(\alpha + \beta)$.
40. $\cos\alpha = -\frac{12}{13}$, quadrant II; $\sin\beta = \frac{1}{2}$, quadrant II. Find $\cos(\alpha - \beta)$.
41. $\sin\alpha = \frac{5}{13}$, quadrant II; $\cos\beta = -\frac{3}{4}$, quadrant III. Find $\tan(\alpha - \beta)$.
42. $\sin\alpha = -\frac{4}{5}$, quadrant IV; $\sin\beta = -\frac{1}{5}$, quadrant IV. Find $\sin(\alpha + \beta)$.
43. $\sin\alpha = -\frac{4}{5}$, quadrant IV; $\cos\beta = \frac{15}{17}$, quadrant IV. Find $\cos(\alpha + \beta)$.
44. $\cos\alpha = -\frac{3}{5}$, quadrant II; $\sin\beta = -\frac{8}{17}$, quadrant III. Find $\sin(\alpha - \beta)$.
45. $\sin\alpha = \frac{2}{3}$, quadrant I; $\cos\beta = -\frac{1}{3}$, quadrant III. Find $\cos(\alpha - \beta)$.
46. $\cos\alpha = \dfrac{\sqrt{2}}{2}$, quadrant IV; $\sin\beta = -\dfrac{\sqrt{3}}{2}$, quadrant III. Find $\tan(\alpha + \beta)$.
47. $\sin\alpha = \frac{2}{3}$, quadrant I; $\tan\beta = \frac{1}{4}$, quadrant I. Find $\sin(\alpha + \beta)$.
48. $\tan\alpha = \frac{3}{4}$, quadrant III; $\sin\beta = -\frac{4}{5}$, quadrant III. Find $\cos(\alpha - \beta)$.
49. $\cos\alpha = \frac{5}{13}$, quadrant IV; $\tan\beta = -\frac{5}{12}$, quadrant IV. Find $\tan(\alpha - \beta)$.
50. $\cos\alpha = \frac{1}{2}$, quadrant I; $\cos\beta = \dfrac{\sqrt{2}}{2}$, quadrant IV. Find $\sin(\alpha + \beta)$.
51. $\tan\alpha = 2$, quadrant III; $\cos\beta = -\frac{3}{5}$, quadrant II. Find $\cos(\alpha - \beta)$.
52. $\sin\alpha = -\frac{15}{17}$, quadrant III; $\tan\beta = -\frac{3}{4}$, quadrant IV. Find $\tan(\alpha + \beta)$.
53. $\sin\alpha = \frac{2}{3}$, quadrant I; $\sin\beta = -\frac{1}{5}$, quadrant III. Find $\sin(\alpha - \beta)$.
54. $\cos\alpha = -\frac{5}{13}$, quadrant III; $\cos\beta = -\frac{8}{17}$, quadrant III. Find $\cos(\alpha + \beta)$.
55. $\cos\alpha = -\dfrac{\sqrt{2}}{2}$, quadrant II; $\sin\beta = \dfrac{\sqrt{3}}{2}$, quadrant II. Find $\tan(\alpha + \beta)$.
56. $\cos\alpha = -\frac{3}{5}$, quadrant III; $\sin\beta = \frac{1}{3}$, quadrant II. Find $\tan(\alpha - \beta)$.

Use the sum and difference identities to verify the following identities.

57. $\sin(\pi - \theta) = \sin\theta$
58. $\sin(\pi + \theta) = -\sin\theta$
59. $\cos(\pi - \theta) = -\cos\theta$
60. $\cos(\pi + \theta) = -\cos\theta$
61. $\tan(\pi - \theta) = -\tan\theta$
62. $\tan(\pi + \theta) = \tan\theta$
63. Use the sum formula to show that the sine function is 2π-periodic; that is, that $\sin(\theta + 2\pi) = \sin\theta$.
64. Use the sum formula to show that the cosine function is 2π-periodic; that is, that $\cos(\theta + 2\pi) = \cos\theta$.
65. Use the sum formula to show that the tangent function is π-periodic; that is, that $\tan(\theta + \pi) = \tan\theta$.

The following identities are important because they express a product of factors as a sum of terms. Verify each identity.

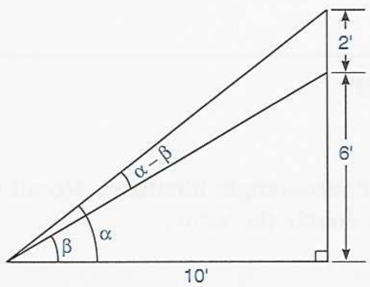
66. $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

67. $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

68. $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$

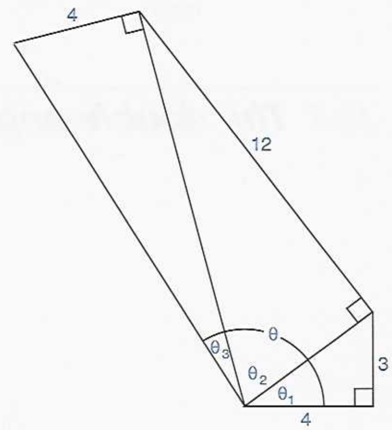
69. $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$

$\boxed{\textbf{70.}}$ A picture on a wall is 2 feet tall and 6 feet above eye level; see the diagram. Compute the exact value of $\sin(\alpha-\beta)$.



71. Referring to the figure, find (a) the exact value of $\tan\alpha$, and (b) use this to find the exact value of x. *Hint:* Compute $\tan(\alpha + 45°)$.

72. Use the identities for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ to find $\sin\theta$ in the diagram.



The following problems are designed to show that the sum and difference identities for sine and cosine [2], [3], and [4], and the cofunction identities, [5] through [10], are true, using the fact that identity [1] is true. The problems are in the necessary logical order.

73. Use identity [1], $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, to verify identity [2], $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. Do this by replacing β by $(-\beta)$ in the identity for $\cos(\alpha+\beta)$ and simplifying, using the even and odd properties for the sine and cosine functions.

74. Verify the identity [5], $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$, by using identity [2], letting $\alpha = \frac{\pi}{2}$ and $\beta = \theta$.

75. The identity [6], $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$, is really the same as identity [5]. Show this as follows. Let $\alpha = \frac{\pi}{2}-\theta$, so that $\theta = \frac{\pi}{2}-\alpha$. Replace $\frac{\pi}{2}-\theta$ by α in the left member of identity [5], then replace θ in the right member by $\frac{\pi}{2}-\alpha$. Then observe that α and θ are arbitrary values, so the result can be rewritten in terms of θ.

$\boxed{\textbf{76.}}$ Verify identity [3], $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, as follows.

$$\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right) \qquad \text{This is identity [5], which we know is true}$$

$$\sin(\alpha+\beta) = \cos\left[\frac{\pi}{2}-(\alpha+\beta)\right] \qquad \text{Replace } \theta \text{ by } \alpha+\beta$$

$$\sin(\alpha+\beta) = \cos\left[\left(\frac{\pi}{2}-\alpha\right)-\beta\right] \qquad \text{Regroup } \frac{\pi}{2}-\alpha-\beta$$

Now use identity [2] to expand the right member of this equation, then apply identities [5] and [6] to simplify the result and obtain identity [3].

77. Use the identity [3], $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, to verify the identity [4], $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$. Do this by replacing β by $(-\beta)$ in identity [3].

78. Verify identity [7], $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$ using the fact that $\tan x = \frac{\sin x}{\cos x}$.

79. Verify identity [8], $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$. See problem 78 for guidance.

80. Verify identity [9], $\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$ using the fact that $\sec x = \frac{1}{\cos x}$.

81. Verify identity [10], $\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$. See problem 80 for guidance.

5–3 The double-angle and half-angle identities

Double-angle identities

Some more important identities are the **double-angle identities.** Recall that if we multiply a value by two we say we double the value.

Double-angle identities

[1]	$\sin 2\alpha = 2\sin\alpha\cos\alpha$	[2-a]	$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
		[2-b]	$\cos 2\alpha = 1 - 2\sin^2\alpha$
[3]	$\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$	[2-c]	$\cos 2\alpha = 2\cos^2\alpha - 1$

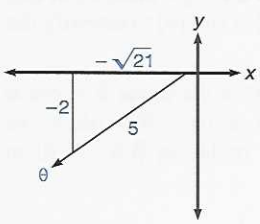
Observe that we present three identities for $\cos 2\alpha$. This is because identities [2-b] and [2-c] get so much use in the development of other identities.

The proof of [1] is as follows.

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \text{Sum identity from section 5–2}$$
$$\sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha \qquad \text{Let } \beta = \alpha$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad \alpha + \alpha = 2\alpha$$

The verification of the remaining identities is left for the exercises. They are done in a similar way, starting with the identities for $\cos(\alpha + \beta)$ and $\tan(\alpha + \beta)$.

■ *Example 5–3 A*



1. If $\sin\theta = -\frac{2}{5}$ and θ lies in quadrant III, find exact values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

First construct a reference triangle for θ to obtain any required trigonometric function values for that angle.

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(-\frac{2}{5}\right)\left(-\frac{\sqrt{21}}{5}\right) \qquad \sin\theta = -\frac{2}{5};\ \cos\theta = -\frac{\sqrt{21}}{5}$$
$$= \frac{4\sqrt{21}}{25}$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \left(-\frac{\sqrt{21}}{5}\right)^2 - \left(-\frac{2}{5}\right)^2 = \frac{21}{25} - \frac{4}{25} = \frac{17}{25}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2\left(\frac{2}{\sqrt{21}}\right)}{1-\left(\frac{2}{\sqrt{21}}\right)^2} = \frac{\frac{4}{\sqrt{21}}}{1-\frac{4}{21}}$$

$$= \frac{\frac{4}{\sqrt{21}}}{\frac{17}{21}} = \frac{4}{\sqrt{21}}\cdot\frac{21}{17} = \frac{4\sqrt{21}}{17}$$

$\tan 2\theta$ could also be obtained from $\tan 2\theta = \dfrac{\sin 2\theta}{\cos 2\theta}$.

2. Find an identity for $\tan 3\theta$ in terms of $\tan\theta$.

$$\tan 3\theta = \tan(2\theta + \theta) \qquad 3\theta = 2\theta + \theta$$

$$= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta} \qquad \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$= \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \frac{2\tan\theta}{1-\tan^2\theta}\tan\theta} \qquad \text{Replace } \tan 2\theta \text{ by } \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2\tan\theta + \tan\theta(1-\tan^2\theta)}{(1-\tan^2\theta) - 2\tan^2\theta} \qquad \text{Multiply numerator and denominator by } (1-\tan^2\theta)$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \qquad \text{Combine}$$

Half-angle identities

A further set of important identities is the **half-angle identities.**

Half-angle identities

[4] $\sin\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$

[5] $\cos\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$

[6-a] $\tan\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$

[6-b] $\tan\dfrac{\alpha}{2} = \dfrac{\sin\alpha}{1+\cos\alpha}$

[6-c] $\tan\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{\sin\alpha}$

We verify identity [5] as follows:

$$2\cos^2\theta - 1 = \cos 2\theta \qquad \text{Identity [2-c] above}$$
$$2\cos^2\theta = 1 + \cos 2\theta$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} \qquad \text{Replace } \theta \text{ by } \frac{\alpha}{2}$$
$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \text{Take the square root of each member}$$

The verification of the remaining identities is left for the exercises.

The choice of plus or minus in identities [4], [5], and [6-a] depends on the quadrant in which the angle in question terminates (using the ASTC rule from section 2–3). It is only possible to determine the quadrant if we have information about the measure of the angle. To see why, consider figure 5–1.

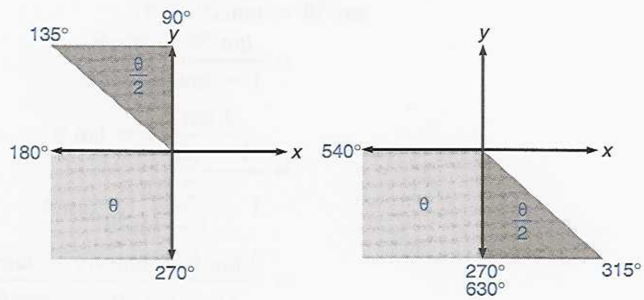


Figure 5–1

If $180° \le \theta \le 270°$ then $90° \le \frac{\theta}{2} \le 135°$; in this case, θ terminates in quadrant III and $\frac{\theta}{2}$ terminates in quadrant II. However, if $540° \le \theta \le 630°$, then $270° \le \frac{\theta}{2} \le 315°$. In this case, θ also terminates in quadrant III but $\frac{\theta}{2}$ terminates in quadrant IV.

These identities have applications such as those shown in example 5–3 B.

■ ***Example 5–3 B***

1. Use the fact that 22.5° is one half of 45° to find the exact value of sin 22.5°.

$$\sin 22.5° = \sin\frac{45°}{2}$$

$$= \pm\sqrt{\frac{1-\cos 45°}{2}} \qquad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}};\ \alpha = 45°$$

$$= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} \qquad \text{We know } \sin 22.5° > 0\text{, so choose plus}$$

$$= \sqrt{\frac{2-\sqrt{2}}{4}} \qquad \frac{1}{2}\left(1-\frac{\sqrt{2}}{2}\right) = \frac{1}{2}\left(\frac{2-\sqrt{2}}{2}\right).$$

$$= \frac{\sqrt{2-\sqrt{2}}}{2}$$

2. $\cos\theta = \frac{3}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Find the exact value for $\cos\frac{\theta}{2}$.

Since $\frac{3\pi}{2} < \theta < 2\pi$, $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$, so $\frac{\theta}{2}$ terminates in quadrant II, where the cosine function is negative.

$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} \qquad \text{Choose minus since } \frac{\theta}{2} \text{ terminates in quadrant II where } \cos\frac{\theta}{2} < 0$$

$$= -\sqrt{\frac{1+\frac{3}{5}}{2}} \qquad \text{Replace } \cos\theta \text{ with } \frac{3}{5}$$

$$= -\sqrt{\tfrac{1}{2}\cdot\tfrac{8}{5}} = -\sqrt{\tfrac{4}{5}} = -\sqrt{\tfrac{20}{25}}$$

$$= -\frac{2\sqrt{5}}{5}$$

■

It is important to understand how to rewrite identities with different forms of the argument. For example, the following identities are all the same; the argument of each is shown in different forms.

[1]	$\sin 2\alpha = 2\sin\alpha\cos\alpha$	Identity [1] of the double-angle identities
	$\sin 4\alpha = 2\sin 2\alpha\cos 2\alpha$	Replace α in [1] by 2α
	$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$	Replace α in [1] by $\frac{\alpha}{2}$

Example 5–3 C illustrates.

■ Example 5–3 C

Rewrite each expression as an expression of the form $a \sin x$, $a \cos x$, or $a \tan x$, for appropriate values of a and x.

1. $2 \sin 4\theta \cos 4\theta$

Compare

$$\begin{array}{ll} [1] & 2 \sin \alpha \cos \alpha = \sin 2\alpha \\ & 2 \sin 4\theta \cos 4\theta \end{array}$$

We can see that we should replace α by 4θ in identity [1] to obtain $2 \sin 4\theta \cos 4\theta$. Then,

$$\begin{aligned} 2 \sin \alpha \cos \alpha &= \sin 2\alpha && \text{Identity [1]} \\ 2 \sin 4\theta \cos 4\theta &= \sin 2(4\theta) && \text{Replace by } 4\theta \\ &= \sin 8\theta \end{aligned}$$

Thus, $2 \sin 4\theta \cos 4\theta = \sin 8\theta$.

2. $\dfrac{4 \tan 2\theta}{1 - \tan^2 2\theta}$

$$\begin{aligned} \frac{2 \tan \alpha}{1 - \tan^2\alpha} &= \tan 2\alpha && \text{Identity [3]} \\ \frac{4 \tan \alpha}{1 - \tan^4\alpha} &= 2 \tan 2\alpha && \text{Multiply each member by 2} \\ \frac{4 \tan 2\theta}{1 - \tan^2 2\theta} &= 2 \tan 4\theta && \text{Replace } \alpha \text{ by } 2\theta \end{aligned}$$

3. $\cos^2 80° - \sin^2 80°$

Compare

$$\begin{aligned} \cos^2\alpha - \sin^2\alpha &= \cos 2\alpha && \text{Identity [2] of the double-angle identities} \\ \cos^2 80° - \sin^2 80° \end{aligned}$$

Since $80°$ replaces α, we know that $\cos 2\alpha$ becomes $\cos 2(80)° = \cos 160°$. Thus, $\cos^2 80° - \sin^2 80° = \cos 160°$. ■

A similar idea is illustrated in example 5–3 D.

■ Example 5–3 D

Find a value of θ for which the statement $\sin 110° = 2 \sin \theta \cos \theta$ is true, then rewrite the statement replacing θ by this value.

Compare

$$\begin{aligned} \sin 110° &= 2 \sin \theta \cos \theta \\ \sin 2\theta &= 2 \sin \theta \cos \theta \end{aligned}$$

Let $2\theta = 110°$, so $\theta = 55°$.
The statement becomes $\sin 110° = 2 \sin 55° \cos 55°$. ■

The identities of this and the previous sections may be combined to verify new identities.

■ ***Example 5–3 E***

Verify the following identities.

1. $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

It is best to work with the right member since it is more complicated.

$$\begin{aligned}
\frac{2\tan\theta}{1+\tan^2\theta} &= 2\cdot\frac{\tan\theta}{\sec^2\theta} && \tan^2\alpha + 1 = \sec^2\alpha \\
&= 2\cdot\tan\theta\cos^2\theta && \cos\alpha = \frac{1}{\sec\alpha} \\
&= 2\cdot\frac{\sin\theta}{\cos\theta}\cos^2\theta \\
&= 2\sin\theta\cos\theta && \frac{1}{\cos\theta}\cdot\cos^2\theta = \cos\theta \\
&= \sin 2\theta && \sin 2\alpha = 2\sin\alpha\cos\alpha
\end{aligned}$$

2. $\sin 4\theta = 8\sin\theta\cos^3\theta - 4\sin\theta\cos\theta$

Although the right member is more complicated, it is easier to begin with $\sin 4\theta$ and expand this expression.

$$\begin{aligned}
\sin 4\theta &= \sin[2(2\theta)] \\
&= 2\sin 2\theta\cos 2\theta && \text{Use } \sin 2\alpha = 2\sin\alpha\cos\alpha\text{, with } \alpha = 2\theta \\
&= 2(2\sin\theta\cos\theta)(2\cos^2\theta - 1) \\
&= 4\sin\theta\cos\theta(2\cos^2\theta - 1) \\
&= 8\sin\theta\cos^3\theta - 4\sin\theta\cos\theta
\end{aligned}$$

Mastery points

Can you

- Write the double-angle and half-angle identities?
- Use the double-angle and half-angle identities to find exact values of $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$, $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$, $\tan\frac{\theta}{2}$?
- Use the double-angle and half-angle identities to derive new identities and to verify given identities?
- Rewrite certain identities as a trigonometric function of $k\theta$, k an integer?

Exercise 5–3

Use the identities of this section to rewrite each expression as an expression of the form $a \sin x$, $a \cos x$, or $a \tan x$, for appropriate values of a and x.

1. $2 \sin \frac{\pi}{4} \cos \frac{\pi}{4}$
2. $2 \sin 52° \cos 52°$
3. $\cos^2 3\pi - \sin^2 3\pi$
4. $2 \cos^2 5\pi - 1$
5. $1 - 2 \sin^2 \frac{\pi}{10}$
6. $\dfrac{2 \tan \frac{\pi}{6}}{1 - \tan^2 \frac{\pi}{6}}$
7. $\dfrac{6 \tan 10°}{1 - \tan^2 10°}$
8. $8 \cos^2 \frac{\pi}{2} - 4$
9. $2 \sin 6\theta \cos 6\theta$
10. $4 \sin 2\theta \cos 2\theta$
11. $6 \cos^2 5\theta - 3$
12. $8 \cos^2 3\theta - 4$
13. $\dfrac{10 \tan 3\theta}{1 - \tan^2 3\theta}$
14. $\dfrac{8 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$
15. $2 - 4 \sin^2 7\theta$
16. $\frac{1}{2} - \sin^2 2\theta$
17. $3 \cos^2 3\theta - 3 \sin^2 3\theta$
18. $2 \cos^2 \frac{\theta}{2} - 2 \sin^2 \frac{\theta}{2}$

Find a value of θ for which each statement is true.

19. $\sin 140° = 2 \sin \theta \cos \theta$
20. $\sin \theta \cos \theta = \frac{1}{2} \sin \frac{\pi}{5}$
21. $\cos \frac{5\pi}{6} = \cos^2 \theta - \sin^2 \theta$
22. $2 \tan 86° = \dfrac{4 \tan \theta}{1 - \tan^2 \theta}$
23. $3 \cos 70° = 6 \cos^2 \theta - 3$
24. $\cos 560° = 1 - 2 \sin^2 \theta$
25. $\sin 10° = \sqrt{\dfrac{1 - \cos \theta}{2}}$
26. $\tan \theta = \sqrt{\dfrac{1 - \cos 46°}{1 + \cos 46°}}$
27. $\cos \theta = \sqrt{\dfrac{1}{2}\left(1 + \cos \dfrac{\pi}{4}\right)}$
28. $\sin \frac{\pi}{6} = \sqrt{\frac{1}{2}(1 - \cos \theta)}$
29. $\tan \frac{2\pi}{5} = \dfrac{1 - \cos \theta}{\sin \theta}$
30. $\cos 40° = \sqrt{\dfrac{1 + \cos \theta}{2}}$

Find the exact value of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$ for each of the following.

31. $\sin \theta = \frac{3}{5}, 0 < \theta < \frac{\pi}{2}$
32. $\sin \theta = -\frac{12}{13}, \pi < \theta < \frac{3\pi}{2}$
33. $\cos \theta = -\frac{4}{5}, \frac{\pi}{2} < \theta < \pi$
34. $\tan \theta = -\frac{3}{4}, \frac{3\pi}{2} < \theta < 2\pi$
35. $\csc \theta = -\frac{8}{5}, \pi < \theta < \frac{3\pi}{2}$
36. $\tan \theta = \frac{5}{12}, \pi < \theta < \frac{3\pi}{2}$

Find the exact value of $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$, and $\tan \frac{\theta}{2}$ for each of the following.

37. $\sec \theta = -\frac{5}{2}, \pi < \theta < \frac{3\pi}{2}$
38. $\tan \theta = -\sqrt{15}, \frac{\pi}{2} < \theta < \pi$
39. $\cot \theta = -2, \frac{3\pi}{2} < \theta < 2\pi$
40. $\cos \theta = \frac{1}{4}, \frac{3\pi}{2} < \theta < 2\pi$

Use the half-angle identities to find the exact value of $\sin \theta$, $\cos \theta$, and $\tan \theta$ for the following values of θ.

41. $15°$, or $\frac{\pi}{12}$
42. $22.5°$, or $\frac{\pi}{8}$
43. $75°$, or $\frac{5\pi}{12}$

Use the sum/difference identities (from section 5–2) and the results of problems 41 and 42 to compute the exact value of the following. Observe that $37.5° = 15° + 22.5°$.

44. $\sin 37.5°$

45. $\cos 37.5°$

46. $\tan 37.5°$

47. Find $\sin 7.5°$; see problem 41.

48. Find $\cos 7.5°$; see problem 41.

Verify the following identities.

49. $\sin 2\theta + 1 = (\sin\theta + \cos\theta)^2$

50. $\cos 2\theta + 2\sin^2\theta = 1$

51. $\cos^4\theta - \sin^4\theta = \cos 2\theta$

52. $\cot\theta = \dfrac{1 + \cos 2\theta}{\sin 2\theta}$

53. $\dfrac{1 + \cos 2\theta}{1 - \cos 2\theta} = \cot^2\theta$

54. $\tan 2\theta = \dfrac{2\tan\theta}{2 - \sec^2\theta}$

55. $\cot\theta - \tan\theta = \dfrac{2\cos 2\theta}{\sin 2\theta}$

56. $2\csc 2\theta = \tan\theta + \cot\theta$

57. $\sin 2\theta - 4\sin^3\theta\cos\theta = \sin 2\theta\cos 2\theta$

58. $\cos 4\theta = 1 - 8\sin^2\theta\cos^2\theta$

59. $\csc^2\theta = \dfrac{2}{1 - \cos 2\theta}$

60. $\dfrac{2\cos^3\theta}{1 - \sin\theta} = 2\cos\theta + \sin 2\theta$

61. $\tan 2\theta = \dfrac{2(\tan\theta + \tan^3\theta)}{1 - \tan^4\theta}$

62. $\cot 4\theta = \dfrac{1 - \tan^2 2\theta}{2\tan 2\theta}$

63. $2\csc 2\theta\sin\theta\cos\theta = 1$

64. $\sec 2\theta = \dfrac{1}{1 - 2\sin^2\theta}$

65. $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$

66. $\dfrac{\csc\theta - \cot\theta}{1 + \cos\theta} = \csc\theta\tan^2\dfrac{\theta}{2}$

67. $\cos^2\dfrac{\theta}{2} = \dfrac{1 - \cos^2\theta}{2 - 2\cos\theta}$

68. $\sec^2\theta - \cos^2\dfrac{\theta}{2} = \tan^2\theta + \sin^2\dfrac{\theta}{2}$

69. $\cos^2\dfrac{\theta}{2}\sin^2\dfrac{\theta}{2} = \dfrac{\sin^2\theta}{4}$

70. $\sin^2\dfrac{\theta}{2} - \cos^2\dfrac{\theta}{2} = -\cos\theta$

71. $\tan^2\dfrac{\theta}{2} + \cos^2\dfrac{\theta}{2} = \dfrac{\cos^2\theta + 3}{2 + 2\cos\theta}$

72. $\tan^2\dfrac{\theta}{2} = \dfrac{2}{1 + \cos\theta} - 1$

73. $\sin^2\dfrac{\theta}{2} = \dfrac{\csc\theta - \cot\theta}{2\csc\theta}$

74. $4\sin^2\dfrac{\theta}{2}\cos^2\dfrac{\theta}{2} = \sin^2\theta$

75. $\dfrac{1 + \sec\theta}{\sec\theta} = 2\cos^2\dfrac{\theta}{2}$

76. $\tan\dfrac{\theta}{2} + \cot\dfrac{\theta}{2} = \dfrac{2}{\sin\theta}$

77. $2\cos^2\dfrac{\theta}{2} - \cos\theta = 1$

78. Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

79. Find an identity for $\cos 3\theta$ in terms of $\cos\theta$. See problem 78.

80. Find identities for (a) $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$ and for (b) $\cos 4\theta$ in terms of $\cos\theta$.

81. Find identities for (a) $\sin 5\theta$ in terms of $\sin\theta$ and for (b) $\cos 5\theta$ in terms of $\cos\theta$.

82. Finding the center of gravity of a certain solid involves the expression $\frac{3}{16}a\left(\dfrac{1 - \cos 2\alpha}{1 - \cos\alpha}\right)$. Show that this is equivalent to $\frac{3}{8}a(1 + \cos\alpha)$.

83. Show that $\tan\dfrac{\alpha}{2} = \dfrac{1 - \cos\alpha}{\sin\alpha}$ (identity [6-c]). Do this as follows. Let $\dfrac{\alpha}{2} = \theta$, so that $\alpha = 2\theta$. Replace $\dfrac{\alpha}{2}$ and α in the identity. Then, simplify the right member; the most direct route will use $\cos 2\theta = 1 - 2\sin^2\theta$.

84. Show that $\tan\dfrac{\alpha}{2} = \dfrac{\sin\alpha}{1 + \cos\alpha}$ (identity [6-b]). See the previous problem.

85. **a.** Use the identity for $\sin\dfrac{\theta}{2}$ with $\theta = 30°$ to find the exact value of $\sin 15°$.

b. Use $\alpha = 45°$, $\beta = 30°$ and $\sin(\alpha - \beta)$ to find the exact value of $\sin 15°$.

c. Show that the values in (a) and (b) are the same. You may find useful the principle that if $a > 0$ and $b > 0$, then $a^2 = b^2$ implies that $a = b$.

86. **a.** Find $\tan 15°$ with the identity for $\tan\dfrac{\alpha}{2}$ (half-angle identity [6-a]), with $\alpha = 30°$.

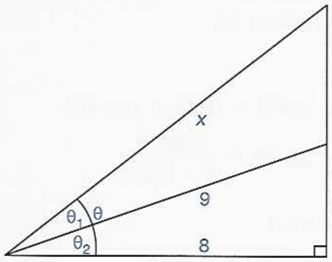
b. Rewrite $\tan 15°$ as $\dfrac{\sin 15°}{\cos 15°}$ and use identities [4] and [5], with $\alpha = 30°$ to compute $\tan 15°$.

c. Show that the values in parts a and b are the same.

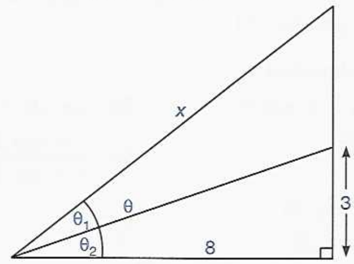
87. Verify half-angle identities [5] and [6-a].

88. Verify the double-angle identities [2-a], [2-b], [2-c], and [3].

89. In the figure, $\theta_1 = \theta_2$. Use the identity for $\cos \frac{\theta}{2}$ to find the length of side x.



90. In the figure, $\theta_1 = \theta_2$. Use the identity for $\tan \frac{\theta}{2}$ to find the length of side x.



The following four identities are important in some situations because they relate the sums and differences of trigonometric expressions to the products of trigonometric expressions. Verify each identity.

91. $\sin 2\alpha + \sin 2\beta = 2 \sin(\alpha + \beta) \cdot \cos(\alpha - \beta)$

92. $\sin 2\alpha - \sin 2\beta = 2 \sin(\alpha - \beta) \cdot \cos(\alpha + \beta)$

93. $\cos 2\alpha + \cos 2\beta = 2 \cos(\alpha + \beta) \cdot \cos(\alpha - \beta)$
(*Hint:* Convert everything to cosine.)

94. $\cos 2\alpha - \cos 2\beta = -2 \sin(\alpha + \beta) \cdot \sin(\alpha - \beta)$
(*Hint:* Convert everything to cosine.)

95. Professor Gilbert Strang of the Massachusetts Institute of Technology has shown[1] an interesting relationship between the identity $\cot 2\theta = \frac{1}{2}\left(\cot \theta - \frac{1}{\cot \theta}\right)$ and the subject of chaotic behavior in iterative systems. Verify this identity.

[1] "A Chaotic Search for *i*," *The College Mathematics Journal*, Vol. 22, No. 1, January 1991.

5–4 Conditional trigonometric equations

Conditional trigonometric equations were introduced in section 1–4, and were revisited several times in chapter 2, as well as in section 5–0. In this section, we examine these equations in a more general way, and examine using the graphing calculator to find approximate solutions.

Remember that whenever we compute an inverse trigonometric function to solve an equation we *use the absolute value of the argument,* which gives us *the reference angle* of the answer.

Primary solutions

In this section we solve for values that are in both degree and radian measure. We determine all solutions that fall between $0° \le x < 360°$ or, in radian measure, $0 \le x < 2\pi$. We call such solutions **primary solutions.** Example 5–4 A illustrates.

■ *Example 5–4 A*

Find all primary solutions for the following trigonometric equations. Find the solutions in degrees (nearest tenth) and radians (four decimal places).

1. $5 \sin \alpha = -2$

$5 \sin \alpha = -2$
$\sin \alpha = -\frac{2}{5}$

Since $\sin \alpha < 0$ all solutions are in quadrants III and IV.

$\alpha' = \sin^{-1}\frac{2}{5}$ — As noted earlier we use $\left|-\frac{2}{5}\right|$
$\alpha' \approx 23.6°$ or 0.4115 radians — Degree mode, radian mode

Degrees:

$$\alpha \approx 180° + 23.6° \text{ or } 360° - 23.6°$$

Radians:

$$\alpha \approx \pi + 0.4115 \text{ or } 2\pi - 0.4115$$

Thus, in degrees $\alpha \approx 203.6°$ or $336.4°$; in radians $\alpha \approx 3.5531$ or 5.8717.

2. $\cos \theta = -\frac{1}{2}$

$\cos \theta = -\frac{1}{2}$ — $\cos \theta < 0$ so all solutions are in quadrants II and III
$\theta' = \cos^{-1}\frac{1}{2}$ — Use the absolute value of $-\frac{1}{2}$
$\theta' = 60°$ or $\frac{\pi}{3}$ radians — Exact values, obtained from the unit circle, figure 2–18
$\theta = 180° \pm 60°$ or $\pi \pm \frac{\pi}{3}$ — In quadrant II $\theta = 180° - \theta'$; in quadrant III $\theta = 180° + \theta'$

$\theta = 120°$ or $240°$ (degrees) or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ (radians).

3. $2 \cos^2\theta - \cos \theta - 1 = 0$

The left member is quadratic in the variable $\cos \theta$. It can be factored. If this is difficult to see, try substitution as illustrated in section 5–0.

$2 \cos^2\theta - \cos \theta - 1 = 0$
$(2 \cos \theta + 1)(\cos \theta - 1) = 0$
$2 \cos \theta + 1 = 0$ or $\cos \theta - 1 = 0$ — Zero factor property

$\cos \theta = -\frac{1}{2}$	$\cos \theta = 1$
$\theta' = \cos^{-1}\frac{1}{2}$	$\theta' = \cos^{-1}1$
$\theta' = 60°$ or $\frac{\pi}{3}$	$\theta' = 0°$ or 0 (radians)

$$\theta = 120° \text{ or } 240° \text{ (degrees)}$$
$$\theta = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \text{ (radians)}; \theta = 0° \text{ or } 0.$$

In degrees, the solutions are 0°, 120°, and 240° and in radians they are 0, $\frac{2\pi}{3}$, and $\frac{4\pi}{3}$.

4. $\tan^2 x + 4\tan x = 1$

$\tan^2 x + 4\tan x - 1 = 0$

This is quadratic, but it will not factor. Solve it using the quadratic formula, as presented in section 5–0.

$$\tan x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2(1)}$$

$a = 1, b = 4, c = -1$ in $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-4 \pm \sqrt{20}}{2} = \frac{-4 \pm 2\sqrt{5}}{2}$$

$\sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}$

$$= -2 \pm \sqrt{5}$$

$\tan x = -2 + \sqrt{5}$	$\tan x = -2 - \sqrt{5}$
$x' = \tan^{-1}(-2 + \sqrt{5})$ $\approx 13.3°, 0.2318$ (radians) x is in quadrants I or III since $-2 + \sqrt{5} > 0$. $x \approx 13.3°$ or $180° + 13.3°$ ≈ 0.2318 or $\pi + 0.2318$	$x' = \tan^{-1}\lvert -2 - \sqrt{5} \rvert$ $= \tan^{-1}(2 + \sqrt{5})$ $\approx 76.7°, 1.3390$ (radians) x is in quadrants II or IV since $-2 - \sqrt{5} < 0$. $x \approx 180° - 76.7°$ or $360° - 76.7°$ $\approx \pi - 1.3390$ or $2\pi - 1.3390$

Thus, $x \approx 13.3°, 193.3°, 103.3°$, and $283.3°$ or $0.2318, 3.3734, 1.8026$, and 4.9442.

Using a graphing calculator to help solve an equation

How the solutions of an equation relate to its graph

To see the relationship between solving an equation and its graph, consider the equation $3 \sin x = 2$. This is equivalent to solving $3 \sin x - 2 = 0$. Now, consider the function $y = 3 \sin x - 2$. Solving $3 \sin x - 2 = 0$ is equivalent to finding all values of x that make $y = 0$ in the function $y = 3 \sin x - 2$. Figure 5–2 shows the graph of this function for the interval 0 to 2π (graphed with the calculator in radian mode).

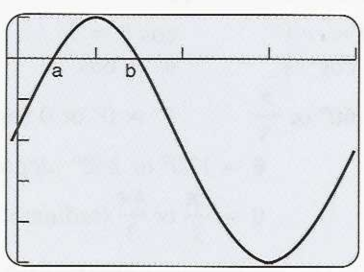


Figure 5–2

$\boxed{\text{Y=}}$ 3 $\boxed{\text{SIN}}$ $\boxed{\text{X|T}}$ $\boxed{-}$ 2, $\boxed{\text{RANGE 0,6.28,1.57,−5,1,1}}$

The x-coordinates of the points where the curve crosses the x-axis, marked a and b, are $x \approx 0.7297$ or 2.4119. Because the value of y is zero at these values of x, we also refer to these x-values as **zeros** of the function.

Thus, if we are solving an equation in which one member is zero, the solutions correspond to the x-intercepts of the graph of the nonzero member of the equation. The graph in figure 5–2 shows that there are two primary solutions.

Using the TI-81 trace function to find approximations to solutions

The Trace function in the calculator can be used to find an approximate value of a solution. For example, graph the function as shown above, then select $\boxed{\text{TRACE}}$. A blinking box appears on the function toward the center of the screen. By using the left and right cursor keys $\boxed{\leftarrow}$ and $\boxed{\rightarrow}$ we can cause the cursor to trace the function, all the while indicating its x- and y-coordinates at the bottom of the screen. By ''tracing'' to the point b we see that x is about 2.286 and y is about 0.057. Of course if we were at the exact solution, y would be zero.

By zooming in (using $\boxed{\text{ZOOM}}$ 2) and reselecting TRACE we can get a better approximation to the actual value, and by repeatedly zooming in again and tracing again we can obtain more and more accurate values for x.

This method of finding approximations to solutions is tedious and inefficient. We next show a much better way to find approximations to solutions.

The TI-81 and Newton's method

There are numeric methods for finding solutions to equations quickly and with great accuracy by using a programmable calculator. One can write a program that searches for a zero of a function. This is useful when the function is well behaved around the zero. For our purposes here, by well behaved we mean that one continuous, smooth line could be used to draw the graph of the function near the zero in question. A good method is called Newton's method. The Texas Instruments TI-81 calculator handbook presents a program called NEWTON that implements this method.

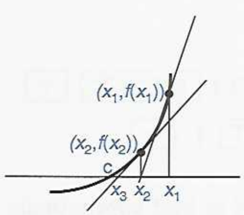


Figure 5–3

Figure 5–3 illustrates how Newton's method gets closer and closer to a root. Assume a function f has a zero at c in figure 5–3. Suppose x_1 is a value of x near c. The program uses the line that is tangent to (i.e., just touches) the function f at the point $(x_1, f(x_1))$ to locate the point x_2, which is closer to c. The program then uses the line that is tangent to the function f at the point $(x_2, f(x_2))$ to locate the point x_3, which is even closer to c. The program repeats this until the difference between the last x-value and the newest x-value is less than a predetermined error value.

The algebraic way in which the program discovers the tangent line at each step is left for a course in the calculus. With a little background in this subject, it is not hard to understand. The program NEWTON can be entered into the calculator as follows:

PRGM ▶ ENTER	Program edit mode
A-LOCK	2nd ALPHA (Alpha lock)

Enter the keys that correspond to the word N E W T O N. For example, T is over the 4 key.

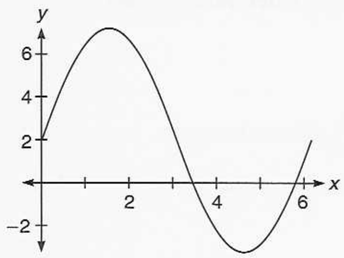
ENTER	Use this after entering the name NEWTON.

Now type in the program as shown.

Program	**Keystroke guide**
:(Xmax − Xmin)/100→D	Xmax is in VARS RNG. Xmin is in VARS RNG. →D is STO▶ x^{-1}.
:Lbl 1	Lbl is in PRGM CTL.
:X-Y_1/NDeriv(Y_1,D)→R	Y_1 is in Y-VARS. NDeriv is MATH 8. ",D" is ALPHA . ALPHA x^{-1}. →R is STO▶ ×.
:If abs (X-R)≤abs (X/1E10)	If is in PRGM CTL. ≤ is in TEST (2nd MATH). 1E10 is 1 EE 10.
:Goto 2	Goto is in PRGM CTL.
:R→X	Use ALPHA × STO▶ X|T.
:Goto 1	
:Lbl 2	
:Disp "ROOT="	Use PRGM I/O 1 A-LOCK + R O O T TEST 1 +
:Disp R	

Example 5–4 B illustrates using this program NEWTON to find approximate solutions to a trigonometric equation.

■ *Example 5–4 B*



Solve the following problems using the programmable calculator.

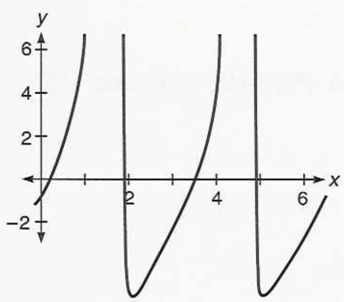
1. $5 \sin \alpha = -2$

This is equivalent to $5 \sin \alpha + 2 = 0$; find the zeros of the function $y = 5 \sin x + 2$.

Graph $y = 5 \sin x + 2$. This is shown above, with Xmin=−.1, Xmax=6.3, Ymin=−3, Ymax=7. *The calculator is in radian mode.* Use the trace feature to position the cursor near the zero between 3 and 4. Now execute the program NEWTON. To do this, select $\boxed{\text{PRGM}}$, select the number that corresponds to the NEWTON program, and use $\boxed{\text{ENTER}}$ to execute the program. The value 3.5531095 appears. This is one of the zeros.

Graph the function again, select trace, position the cursor near the second zero, and run the program NEWTON again. The value 5.871668461 appears. This is an approximation to the second zero.

Repeating these steps in degree mode will find approximations to the zeros in degree mode. Of course Xmin should be something like $-10°$, and Xmax about 360°. The results displayed, in degrees, are 203.5781785° and 336.4218215°.



2. $\tan^2 x + 4 \tan x = 1$

This is equivalent to $\tan^2 x + 4 \tan x - 1 = 0$. Graph $y = \tan^2 x + 4 \tan x - 1$ with Xmin=−0.1, Xmax=7, Ymin=−3, Ymax=6.3. This would be entered as Y_1=(tan X)^2+4tan X−1. The graph is shown in the figure.

Select trace, position the cursor near the first zero, and run the program NEWTON. The value 0.2318238045 appears. This is an approximation to the first zero.

Regraph the function and repeat the first step at each zero. The values that appears are 1.802620131, 3.373416458, and 4.944212785.

As in part 1 of the example, redo the problem in degree mode to obtain the results in degrees. The values displayed are 13.28252559°, 103.2825256°, 193.2825256°, and 283.2825256°. ■

Using identities to help solve an equation

When an expression involves more than one trigonometric function we often use identities to rewrite the equation in terms of a single trigonometric function. This is illustrated in example 5–4 C.

■ ***Example 5–4 C***

Find all primary solutions for the following trigonometric equations. Find the solutions in degrees (nearest tenth) and radians (four decimal places).

1. $\tan\theta - \cot\theta = 0$

$$\tan\theta - \cot\theta = 0$$
$$\tan\theta - \frac{1}{\tan\theta} = 0 \qquad \cot\theta = \frac{1}{\tan\theta} \text{ where } \tan\theta \neq 0$$
$$\tan^2\theta - 1 = 0 \qquad \text{Multiply each term by } \tan\theta$$
$$\tan^2\theta = 1$$
$$\tan\theta = \pm 1$$

When $\tan\theta = 1$, θ' is $45°$ or $\frac{\pi}{4}$ (see table 2–1), so using this fact and the ASTC rule for $\tan\theta > 0$ we obtain $\theta = 45°$ or $180° + 45°$, or $\frac{\pi}{4}$ or $\pi + \frac{\pi}{4}$.

When $\tan\theta = -1$, $\theta' = 45°$ or $\frac{\pi}{4}$, but $\theta = 180° - 45°$ or $360° - 45°$, or in radians $\pi - \frac{\pi}{4}$ or $2\pi - \frac{\pi}{4}$.

Thus, the primary solutions in degrees are 45°, 135°, 225°, and 315° and in radians are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

2. $2\cos^2 x - 3\sin x - 3 = 0$

$$2\cos^2 x - 3\sin x - 3 = 0$$
$$2(1 - \sin^2 x) - 3\sin x - 3 = 0 \qquad \cos^2\theta = 1 - \sin^2\theta$$
$$2 - 2\sin^2 x - 3\sin x - 3 = 0$$
$$-2\sin^2 x - 3\sin x - 1 = 0$$
$$2\sin^2 x + 3\sin x + 1 = 0$$
$$(2\sin x + 1)(\sin x + 1) = 0$$

$2\sin x + 1 = 0$	$\sin x + 1 = 0$
$2\sin x = -1$	$\sin x = -1$
$\sin x = -\frac{1}{2}$	$x = 270°$ or $\frac{3\pi}{2}$
$x = 210°, 330°$, or $\frac{7\pi}{6}, \frac{11\pi}{6}$	

The primary solutions are 210°, 270°, 330° or $\frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$. ■

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Finding all solutions to a trigonometric equation

We need to take note of the fact that there are an infinite number of solutions to the equations we solved in the preceding problems. Because the trigonometric functions are periodic the set of all solutions can be found by adding all integral values of the appropriate period (2π or π) to the solution.

This can be illustrated for part 3 of example 5–4 A, where we found that the primary solutions to $\cos \theta = -\frac{1}{2}$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ (in radians). However, the cosine function is 2π-periodic, which means that $\cos(\theta + 2k\pi) = \cos \theta$ for any value of θ and for integer values of k. Thus, the actual set of all radian-valued solutions for this problem is $\frac{2\pi}{3} + 2k\pi$ and $\frac{4\pi}{3} + 2k\pi$, k any integer. This idea is illustrated in figure 5–4.

We use this periodicity for solving trigonometric equations where the coefficient of the argument is not 1. In this situation, it is easier to find all solutions than to find just the primary solutions. This is illustrated in example 5–4 D.

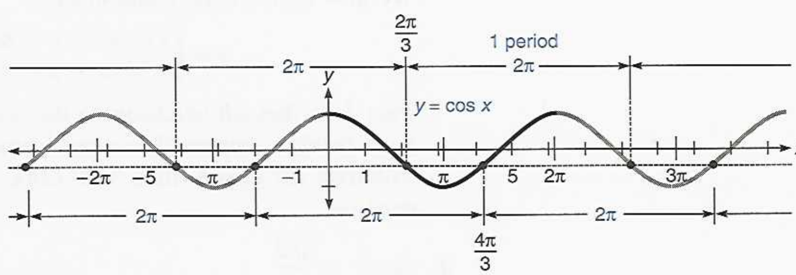


Figure 5–4

■ ***Example 5–4 D***

Find *all* solutions for the following trigonometric equations. Find the solutions in degrees (nearest tenth) and radians (four decimal places).

1. $3 \sin 2x = -2$

$$3 \sin 2x = -2$$
$$\sin 2x = -\tfrac{2}{3}$$

Since $\sin 2x < 0$, $2x$ is in quadrants III or IV.

$$(2x)' = \sin^{-1}\tfrac{2}{3}$$
$$(2x)' \approx 41.8° \text{ or } 0.7297 \text{ radians}$$

Note 1. Do not divide both members by 2 at this point. It is necessary to find all solutions for $2x$ *before* dividing by 2.

2. Although we show the intermediate values above as 41.8° and 0.7297, it is important to keep the maximum accuracy of the calculator up to the last step of the problem. Thus, the calculations that follow are actually performed with the values $\boxed{41.8103149}$ and $\boxed{0.7297276562}$. All calculators have the capability to store at least one value in memory, which should be used to avoid tedious and error-prone reentry of values.

$$2x \approx \begin{cases} 180° + 41.8° \text{ or } 360° - 41.8° \text{ (degrees)} \\ \pi + 0.7297 \text{ or } 2\pi - 0.7297 \text{ (radians)} \end{cases}$$

$2x$ is an angle in quadrants III or IV

$$2x \approx \begin{cases} 221.8°, 318.2° \text{ (degrees)} \\ 3.8713, 5.5535 \text{ (radians)} \end{cases}$$

Primary solutions for $2x$

To describe all solutions we add multiples of the period of the sine function, 360° or 2π.

$$2x \approx \begin{cases} 221.8° + k \cdot 360°, 318.2° + k \cdot 360° \\ 3.8713 + 2k\pi, 5.5535 + 2k\pi \end{cases}$$

We now divide each solution by 2.

$$x \approx \begin{cases} 110.9° + k \cdot 180°, 159.1° + k \cdot 180° \\ 1.9357 + k\pi, 2.7768 + k\pi \end{cases}$$

This describes *all* solutions to the equation. To find primary solutions for x we would compute the values above for $k = 0$ and $k = 1$. If $k = 2$ the solutions are greater than 360° (2π), and if k is negative the solutions are negative.

2. $\tan \frac{x}{2} = \frac{\sqrt{3}}{3}$

$$\tan \frac{x}{2} = \frac{\sqrt{3}}{3}$$

$$\left(\frac{x}{2}\right)' = \tan^{-1}\frac{\sqrt{3}}{3}$$

$$\left(\frac{x}{2}\right)' = 30°, \frac{\pi}{6}$$

$$\frac{x}{2} = 30°, 210°, \text{ or } \frac{\pi}{6}, \frac{7\pi}{6}$$

These are the primary solutions for $\frac{x}{2}$

The tangent function is π-periodic. Thus, we add integer multiples of 180° (π) to obtain all solutions.

$$\frac{x}{2} = \begin{cases} 30° + k \cdot 180°, 210° + k \cdot 180° \text{ (degrees)} \\ \frac{\pi}{6} + k\pi, \frac{7\pi}{6} + k\pi \text{ (radians)} \end{cases}$$

This describes all solutions. However, $210° - 30° = 180°$, and similarly $\frac{7\pi}{6} - \frac{\pi}{6} = \pi$, so the solutions can be described more compactly.

$$\frac{x}{2} = 30° + k \cdot 180°, \text{ or } \frac{\pi}{6} + k\pi$$

$$x = 60° + k \cdot 360° \text{ or } \frac{\pi}{3} + 2k\pi \qquad \text{Multiply each member by 2}$$

All solutions for x are $x = 60° + k \cdot 360°$ or $\frac{\pi}{3} + 2k\pi$. ■

Equations involving more than one multiple of the angle

If an equation mixes multiples of values with the values themselves, such as θ and 2θ in example 5–4 E, we can eliminate the multiple value with an appropriate identity.

■ ***Example 5–4 E***

Solve $\sin 2\theta - \sin\theta = 0$; find primary solutions.

$$\sin 2\theta - \sin\theta = 0$$
$$2\sin\theta\cos\theta - \sin\theta = 0 \qquad \sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\sin\theta(2\cos\theta - 1) = 0$$
$$\sin\theta = 0 \text{ or } 2\cos\theta - 1 = 0$$

$\sin\theta = 0$	$2\cos\theta - 1 = 0$
	$\cos\theta = \frac{1}{2}$
$\theta = 0°, 180°$, or $0, \pi$ (radians)	$\theta = 60°, 300°$, or $\frac{\pi}{3}, \frac{5\pi}{3}$ (radians)

The solutions are $0°, 60°, 180°, 300°$ or $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ (radians). ■

Mastery points

Can you

- Solve linear and quadratic trigonometric equations?
- Solve trigonometric equations involving multiple angles?
- Solve trigonometric equations by applying an appropriate identity?

Exercise 5–4

Find all primary solutions to the following trigonometric equations. Leave answers in both degrees and radians. All answers should be exact.

1. $\tan\theta + 1 = 0$ **2.** $\sin\theta - 1 = 0$ **3.** $2\cos\theta - 1 = 0$ **4.** $2\cos\theta + 1 = 0$
5. $\sqrt{3}\tan\theta - 1 = 0$ **6.** $\cot\theta + \sqrt{3} = 0$ **7.** $\csc\theta + 2 = 0$ **8.** $\sec\theta - 2 = 0$
9. $3\sin^2\theta - 3 = 0$ **10.** $3\csc^2\theta = 3$ **11.** $\sec^2\theta = 1$ **12.** $\tan^2\theta - 1 = 0$

13. $(\cos\theta - 1)(\sin\theta + 1) = 0$
14. $(\sec\theta + 2)(\csc\theta - 2) = 0$
15. $(2\cos^2\theta - 1)(\cot\theta - 1) = 0$
16. $(3\tan^2\theta - 1)(\sqrt{3}\sec\theta - 2) = 0$
17. $\sin^2\theta - \sin\theta = 0$
18. $\cos^2\theta + \cos\theta = 0$
19. $\tan^2\theta - \sqrt{3}\tan\theta = 0$
20. $\cos^2\theta - \frac{1}{2}\cos\theta = 0$
21. $2\sin^2\theta + \sin\theta - 1 = 0$
22. $\cos^2\theta + 2\cos\theta + 1 = 0$
23. $2\sin^3\theta - \sin\theta = 0$
24. $2\cos^2\theta + 3\cos\theta = 2$
25. $2\sin\theta\cos\theta - \sin\theta = 0$
26. $2\sin\theta\cos\theta + \cos\theta = 0$
27. $\sqrt{3}\tan\theta\cot\theta + \cot\theta = 0$
28. $2\tan^2\theta\cos\theta - \tan^2\theta = 0$
29. $\tan x\cot x = 0$
30. $\sin x\cos x = 0$
31. $2\sin x - \csc x + 1 = 0$
32. $2\cos x + \sec x - 3 = 0$
33. $\tan x + \cot x = -2$
34. $2 - \sin x - \csc x = 0$
35. $2\sin^2 x - \cos x = 1$
36. $2\cos^2 x - 3\sin x = 3$
37. $4\tan^2 x = 3\sec^2 x$
38. $4\cot^2 x - 3\csc^2 x = 0$
39. $\sin^2 x - \cos^2 x = 0$
40. $\cot^2 x + \csc^2 x = 0$
41. $2\tan^2 x\sin x = \tan^2 x$
42. $\sin^2 x\cos x - \cos x = 0$

Solve the following equations using the quadratic formula if necessary and the calculator. Find the primary solutions in both radians and degrees. Round radian answers to hundredths, and degree answers to tenths.

43. $6\sin^2 x - 2\sin x - 1 = 0$
44. $3\cos^2 x + \cos x - 2 = 0$
45. $\cot^2 x - 3\cot x - 2 = 0$
46. $\tan^2 x + 5\tan x + 2 = 0$
47. $\sec^2 x - 2\sec x - 4 = 0$
48. $2\csc^2 x - \csc x - 5 = 0$
49. $\tan x + 2\sec x = 3$
50. $3\cot x - \csc x - 1 = 0$

Find all solutions to the following trigonometric equations, both in degrees and in radians.

51. $\cos x = \frac{1}{2}$
52. $\sin x = 1$
53. $\cot x = -\sqrt{3}$
54. $\cos x = -\frac{\sqrt{3}}{2}$
55. $\sin x = -\frac{\sqrt{2}}{2}$
56. $\tan x = -1$
57. $\tan x = 1$
58. $\sec x = \frac{2}{\sqrt{3}}$
59. $\csc x = 2$
60. $\tan\frac{x}{2} = 1$
61. $\sin\frac{x}{2} = \frac{\sqrt{3}}{2}$
62. $\sin 3x = 0$
63. $\cos 3x = -1$
64. $\sec\frac{x}{2} = 1$
65. $3\cot 2x = \sqrt{3}$
66. $2\sin 3x = -1$
67. $2\cos 4x = -1$
68. $-\sqrt{3}\tan 5x = 1$
69. $2\cos 2x + 1 = 0$
70. $\tan 2\theta - 1 = 0$
71. $\cot 2\theta - \sqrt{3} = 0$
72. $2\cos 3\theta = -1$
73. $2\sin 2\theta = 1$
74. $\sin\frac{\theta}{3} = \frac{\sqrt{3}}{2}$
75. $\sec 3\theta = 2$
76. $\csc 2\theta = -\frac{2\sqrt{3}}{3}$
77. $\sqrt{3}\tan\frac{\theta}{4} = 1$
78. $\cot\frac{\theta}{3} = \frac{\sqrt{3}}{3}$

Find the primary solutions to the following trigonometric equations, both in radians and degrees. Find solutions in radians to hundredths, and in degrees to the nearest tenth of a degree, where necessary.

79. $\cos 2\theta + \sin\theta = 0$
80. $\cos 2\theta - \cos\theta = 0$
81. $\sin 2\theta + \sin\theta = 0$
82. $\cos^2\theta - \sin^2\theta = 1$
83. $\cos 2\theta = 1 - \sin\theta$
84. $\cos 2\theta = \cos\theta - 1$
85. $\sin\frac{\theta}{2} = \tan\frac{\theta}{2}$
86. $\sin\frac{\theta}{2} = \cos\theta$
87. $2\sec\theta = \csc^2\frac{\theta}{2}$
88. $\sin^2\frac{\theta}{2} = \cos\theta$
89. $\tan\frac{\theta}{2} = \cos\theta - 1$
90. $\cot\theta - \tan\frac{\theta}{2} = 0$
91. $\sin 2\theta - \cos\theta = \cos^2\theta$

In the mathematical modeling of an aerodynamics problem the following equation arises:

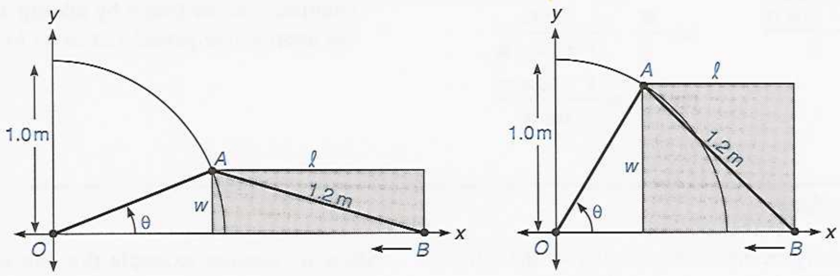
$$y = x\cos A\cos B - x^2\cos A\sin B - x^3\sin A$$

Problems 92 and 93 use this equation.

92. If $A = 0.855$, $B = 1.052$, and $y = 0$, solve for x to the nearest 0.01.

93. If $B = 0.7$, $x = 2$, and $y = -8$, find A to the nearest 0.01. Find the least nonnegative solution(s).

A mechanical device is constructed as shown in the diagram. The arm OA moves through angle θ, from 0° to 90°. Two positions are shown. Point A moves along a circle of radius 1.0 meters, and point B moves horizontally only. The distance AB is fixed by arm AB at 1.2 meters. The area of the shaded rectangle is the product of its length and width, $A = \ell w$.



94. Show that $A = \sin\theta\sqrt{1.44 - \sin^2\theta}$. (The units are square meters.)

95. Find A, to the nearest 0.01 m², when θ = 0°, 30°, 45°, 60°, and 90°.

$\boxed{\textbf{96.}}$ Find θ when $A = 0.5$ m². Round the answer to the nearest 0.1°.

Chapter 5 summary

- **A trigonometric identity** is a trigonometric equation that is true for all permissible replacements of the variable for which each member is defined.
- To verify that a trigonometric equation is an identity we must show that each member of the equation is equivalent to the same expression.
- To show that an equation is not an identity find a value for the variable for which the statement is not true. This value is called a counter example.
- **Reciprocal identities**

$$\csc\theta = \frac{1}{\sin\theta},\quad \sec\theta = \frac{1}{\cos\theta},\quad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\csc\theta},\quad \cos\theta = \frac{1}{\sec\theta},\quad \tan\theta = \frac{1}{\cot\theta}$$

- **Tangent and cotangent identities**

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

- **Fundamental identity of trigonometry**

$$\sin^2\theta + \cos^2\theta = 1$$

Pythagorean identities	**Useful forms**
$\sin^2\theta + \cos^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$
	$\cos^2\theta = 1 - \sin^2\theta$
$\sec^2\theta = \tan^2\theta + 1$	$\tan^2\theta = \sec^2\theta - 1$
	$\sec^2\theta - \tan^2\theta = 1$
$\csc^2\theta = \cot^2\theta + 1$	$\cot^2\theta = \csc^2\theta - 1$
	$\csc^2\theta - \cot^2\theta = 1$

- **Sum and difference identities for sine and cosine**

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

- When the sum of two angles is 90°, or $\frac{\pi}{2}$ radians, the angles are said to be complementary.
- **Cofunction identities**

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$
$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$
$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$$

- **Double-angle identities**

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} \qquad \cos 2\alpha = 1 - 2\sin^2\alpha$$
$$\cos 2\alpha = 2\cos^2\alpha - 1$$

- **Half-angle identities**

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$$

- **Primary solutions** are solutions that fall between $0° \le x \le 360°$ or, in radian measure, $0 \le x < 2\pi$.
- The trigonometric functions are periodic, so the set of all solutions can be found by adding all integral values of the appropriate period (2π or π) to the solution.

Chapter 5 review

[5–1] Show that the trigonometric expression on the left is equivalent to the simplified expression on the right.

1. $\dfrac{\cot\theta}{\cos\theta}$; $\csc\theta$

2. $\sec\theta\tan\theta$; $\sec^2\theta\sin\theta$

3. $\dfrac{\tan^2\theta}{\sec^2\theta-1}$; $\sin^4\theta\csc^4\theta$

4. $\dfrac{\csc^2\theta-1}{\sec^2\theta-1}$; $\cot^4\theta$

5. $\dfrac{\csc\theta\tan\theta}{\sin\theta}$; $\csc\theta\sec\theta$

6. $\sin^2\theta-\cos^2\theta$; $2\sin^2\theta-1$

Obtain an equivalent expression involving only the sine and cosine functions. Simplify the resulting expression.

7. $\csc\theta-\sec\theta$

8. $\tan\theta+\cot\theta$

9. $\dfrac{\sec\theta}{\tan\theta-\cot\theta}$

10. $\dfrac{1-\cot\theta}{\csc\theta+1}$

11. $\dfrac{\sec^2\theta-1}{\sec^2\theta}$

12. $\dfrac{1-\cot^2\theta}{\csc^2\theta-1}$

Verify that each equation is an identity.

13. $\csc x-\tan x\cot x=\csc x-1$

14. $\sin^2x+\sin^2x\cot^2x=1$

15. $\csc^2x-\sec^2x=\csc^2x\sec^2x(\cos^2x-\sin^2x)$

16. $\dfrac{1}{\csc x-\cot x}=\dfrac{\sin x}{1-\cos x}$

17. $\tan x-1=\sec x(\sin x-\cos x)$

18. $\dfrac{\csc^2x-1}{\sin^2x}=\cos^2x\csc^4x$

19. $\dfrac{1}{1+\csc x}+\dfrac{1}{1-\csc x}=-2\tan^2x$

20. $\sin^2x+\sin^2x\cos^2x=1-\cos^4x$

21. $\tan^4x+\tan^2x=\dfrac{\sec^2x}{\cot^2x}$

22. $\dfrac{1-\cot x}{1+\csc x}=\dfrac{\sin x-\cos x}{\sin x+1}$

Show by counter example that the following equations are not identities.

23. $\sin\theta+\cos\theta=1$

24. $\tan\theta-\sin\theta\cos\theta=0$

25. $\dfrac{1}{\cot\theta-\csc\theta}=\sec\theta$

[5–2] Use the sum and difference formulas to find the exact value of the following.

26. $\cos\dfrac{\pi}{12}$

27. $\tan\left(-\dfrac{\pi}{12}\right)$

28. $\sin 105°$

29. $\tan(-15°)$

30. Given $\sin\alpha=\frac{3}{4}$ and $\cos\beta=-\frac{5}{6}$, α and β lie in quadrant II, find
 a. $\sin(\alpha-\beta)$ **b.** $\tan(\alpha+\beta)$

31. Given $\cos\alpha=-\dfrac{\sqrt{3}}{2}$ and $\sin\beta=-\dfrac{1}{4}$, α lies in quadrant II and β lies in quadrant III, find
 a. $\cos(\alpha+\beta)$ **b.** $\tan(\alpha-\beta)$

32. Given $\sin\alpha=-\frac{5}{12}$ and $\cos\beta=\frac{8}{17}$, α lies in quadrant IV and β lies in quadrant I, find
 a. $\sec(\alpha+\beta)$ **b.** $\cot(\alpha-\beta)$

Using the sum and difference formulas, verify each of the following identities.

33. $\dfrac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)}=\dfrac{\cot\alpha+\cot\beta}{\cot\alpha\cot\beta+1}$

34. $\cos\left(\dfrac{3\pi}{2}+\theta\right)=\sin\theta$

35. $\sin\left(\dfrac{\pi}{4}-\theta\right)=\dfrac{\sqrt{2}}{2}(\cos\theta-\sin\theta)$

36. $\dfrac{\cos(\alpha+\beta)}{\sin\alpha\cos\beta}=\cot\alpha-\tan\beta$

37. Using the identity $\cot \alpha = \dfrac{1}{\tan \alpha}$ and the identity $\tan(\alpha + \beta)$, show that $\cot(\theta + \pi) = \cot \theta$.

[5–3] Using the double-angle identities, find angle θ that makes the following statements true.

38. $\cos \theta = \cos^2 62° - \sin^2 62°$

39. $\sin \theta = 2 \sin 5\pi \cos 5\pi$

40. $\tan \theta = \dfrac{2 \tan \frac{7\pi}{12}}{1 - \tan^2 \frac{7\pi}{12}}$

41. $\cos 24° = 1 - 2 \sin^2 \theta$

42. Express $6 \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2}$ as a single trigonometric function of a constant k times θ.

43. Express $\dfrac{4 \tan 4\theta}{1 - \tan^2 4\theta}$ as a single trigonometric function of a constant k times θ.

44. Given $\cos \theta = -\frac{5}{12}$, θ lies in quadrant III, find the exact value of **a.** $\tan 2\theta$ **b.** $\sin 2\theta$

45. Given $\sin \theta = \frac{4}{5}$, θ lies in quadrant II, find the exact value of **a.** $\cos 2\theta$ **b.** $\tan 2\theta$

46. Given $\tan \theta = -\frac{5}{4}$, θ lies in quadrant IV, find the exact value of **a.** $\csc 2\theta$ **b.** $\cot 2\theta$

Verify the following identities.

47. $\sin 2x - \cos x = \cos x(2 \sin x - 1)$

48. $1 + \cos 2x = 2 \cos^2 x$

49. $\dfrac{\cos 2x}{2 - 4 \sin^2 x} = \dfrac{1}{2}$

50. $\tan 2x = \dfrac{2 \cot x}{\cot^2 x - 1}$

51. $\sin 2x - \cos 2x = 2 \cos x(\sin x - \cos x) + 1$

Using the half-angle identities, find the exact value of the following.

52. $\tan 22.5°$

53. $\cos(-15°)$

54. $\sin \dfrac{5\pi}{12}$

55. $\cos \dfrac{3\pi}{8}$

56. Given $\cos x = \frac{12}{13}$, $0 < x < \dfrac{\pi}{2}$, find

a. $\sin \dfrac{x}{2}$ **b.** $\tan \dfrac{x}{2}$

57. Given $\sin x = -\frac{2}{3}$, $\pi < x < \dfrac{3\pi}{2}$, find

a. $\sec \dfrac{x}{2}$ **b.** $\cot \dfrac{x}{2}$

58. Find $\sin \dfrac{\theta}{2}$ if $\cos \theta = -\dfrac{7}{18}$ and $\pi < \theta < \dfrac{3\pi}{2}$.

Verify the following equations are identities.

59. $\cot \dfrac{\theta}{2} = \dfrac{1 + \cos \theta}{\sin \theta}$

60. $\sec^2 \dfrac{\theta}{2} - \tan^2 \dfrac{\theta}{2} = 1$

61. $\tan \dfrac{\theta}{2} \csc^2 \dfrac{\theta}{2} = \dfrac{2}{\sin \theta}$

[5–4] Solve the following conditional equations for $0 \le x \le \dfrac{\pi}{2}$.

62. $2 \sin x - 1 = 0$

63. $3 \cot^2 x - 1 = 0$

64. $(\sin x - 1)(2 \cos x - 1) = 0$

65. $(4 \sin^2 x - 1)(\sec x - 2) = 0$

Solve the following conditional equations for $0° \le \theta < 360°$.

66. $\cot^2 \theta - \cot \theta = 0$

67. $\sec^2 \theta - 4 = 0$

68. $2 \cos^2 \theta - \cos \theta - 1 = 0$

69. $2 \sin \theta - \csc \theta + 1 = 0$

70. $2 \cot \theta \cos \theta = \cot^2 \theta$

71. $2 \sin^2 \theta - 3 \cos \theta = 3$

Find all solutions in radians to the following equations. Use the quadratic formula and calculator where necessary (round such answers to the nearest hundredth).

72. $\cos^2 x - 1 = 0$

73. $\tan^2 x - 3 \tan x - 3 = 0$

74. $\sin x - 2 \csc x = 5$

75. $\sec^2 x - \sec x = 2$

Solve each of the following equations for $0 \le x < 2\pi$ (primary solutions, radians).

76. $2 \sin 4x = 1$

77. $2 \cos \dfrac{x}{2} - \sqrt{3} = 0$

78. $\sqrt{3} \tan \dfrac{x}{3} + 1 = 0$

79. $\sin^2 \dfrac{x}{4} = \dfrac{1}{2}$

Solve each of the following equations for $0° \le 0 < 360°$.

80. $2 \cos 5\theta = 1$

81. $3 \tan^2 \dfrac{\theta}{4} = 9$

82. $\tan 6\theta - \cot 6\theta = 0$

83. $\cos \theta + \sin 2\theta = 0$

Find all solutions in radians to the following equations. Find solutions to the nearest hundredth, where necessary.

84. $\cos \dfrac{x}{2} - \sin x = 0$

85. $(\cot 4x - \sqrt{3})(\csc 3x + 2) = 0$

86. $3 \sin^2 2x - \sin 2x - 2 = 0$

Chapter 5 test

1. Show that the expression $\csc^2 x \sin x \cos x$ is equivalent to $\cot x$.

2. Write the expression $\dfrac{\csc x - \sec x}{\tan x + \cot x}$ as an expression in sine and cosine and simplify.

3. Show by counter example that the equation $\cot x - 2 \tan x = 1$ is not an identity.

4. Given $\cos \alpha = -\frac{1}{2}$ and $\sin \beta = \frac{15}{17}$, α lies in quadrant III and β lies in quadrant II, find $\sin(\alpha + \beta)$.

5. Given $\cos x = -\frac{8}{17}$, x lies in quadrant II, find $\sin 2x$.

6. Given $\csc x = \frac{5}{3}$, $\dfrac{\pi}{2} < x < \pi$, find $\tan \dfrac{x}{2}$.

7. Using the appropriate half-angle formula, find $\sec 22.5°$.

8. Verify the following identities.

   a. $\dfrac{1 + \cot \theta}{\csc \theta} = \sin \theta + \cos \theta$

   b. $\dfrac{\cos^2 x - 1}{\sin^2 x} = -1$

   c. $\cos\left(\theta - \dfrac{3\pi}{2}\right) = -\sin \theta$

   d. $\cos 2x - \sin 2x = -1 + 2 \cos x(\cos x - \sin x)$

9. Solve the equation $4 \cos^2 x - 1 = 0$ for $0 \le x < 2\pi$.

10. Solve the equation $(\cot \theta - \sqrt{3})(\sec \theta + 2) = 0$ for $0° \le 0 < 360°$.

11. Find all primary solutions, in radians, to the equation $6 \sin^2 x + 5 \sin x - 1 = 0$. (Use the quadratic formula and the calculator if necessary.)

12. Solve the equation $\sin^2 5\theta - 1 = 0$ for $0° \le 0 < 360°$.

13. Find all primary solutions, in radians, to the equation $\sec^2 \dfrac{x}{4} - 2 = 0$.

14. If $\sin^2 x = \dfrac{1 + \sqrt{1 - m^2}}{2}$, find $\sin 2x$. (*Hint:* Use $\sin^2 2x = 4 \sin^2 x \cos^2 x$.)

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